A TREATISE

ON

SPHERICAL TRIGONOMETRY,

TOGETHER WITH

A SELECTION OF PROBLEMS AND THEIR SOLUTIONS

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CAMBRIDGE:
PRINTED AT THE UNIVERSITY PRESS,
FOR J. & J. J. DEIGHTON, AND T. STEVENSON, CAMBRIDGE;
J. G. & F. RIVINGTON, LONDON.
M.DCCC.XLI.

SPHERICAL TRIGONOMETRY.

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ERRATA.

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A TREATISE

ON

SPHERICAL TRIGONOMETRY.

1. THE boundary of every plane section of a sphere is a circle.

If the cutting plane pass through the center, this is evident; and in this case the section is called a *great circle*, and is determined when any two points on the surface of the sphere through which it passes are given. All great circles are equal to one another, since they have the same radius, namely that of the sphere; and they all bisect one another, since their planes intersect in diameters of the sphere. Hence the distance of the points of intersection of two great circles measured on the sphere is a semi-circumference.

If the cutting plane does not pass through the center of the sphere, from O (fig. 1.) the center, drop upon it the perpendicular OC, and join C with any point A in the boundary of the section; then

$$AC = \sqrt{AO^2 - OC^2}, \text{ which is invariable;}$$

therefore the boundary of the section is a circle whose center is C; and it is called a *small circle*. Arcs of small circles are very rarely used; and when hereafter an arc of a circle is mentioned, an arc of a great circle, unless the contrary be specified, is invariably intended, and in most cases it is employed to denote the angle which it subtends at the center of the sphere, no regard being had to the radius of the sphere.

2. If OC be produced to meet the surface of the sphere in P and P', then

$$PA = \sqrt{PC^2 + AC^2}, \text{ which is invariable.}$$

Also if PAM be an arc of a great circle passing through P and A, since in equal circles equal straight lines cut off equal arcs, the length of the arc PA is invariable. Therefore the distance of P is the same from every point in the perimeter of the circle AB, whether measured along the straight line, or the arc of a great circle, drawn from it to the point. The point P, and the point P' which has evidently the same property, are called the nearer and more remote *poles* of the circle AB; being the extremities of that diameter of the sphere which is perpendicular to the plane of the circle. They are also the poles of all circles of the sphere whose planes are parallel to ACB.

If MN be the *great* circle of which P is the pole, since OP is perpendicular to the plane MON, and the angle POM is consequently a right angle, the distance of P from every point in the boundary of MN, measured on a great circle, is a quadrant.

3. The angle at which two arcs of great circles intersect on the surface of the sphere (in the same way as for any other curves) is the angle between their tangents at the point of intersection, and, consequently, is the same as the angle between the planes in which the arcs lie; for, as the tangents are situated in the same planes, respectively, with the arcs, and are perpendicular to the radius of the sphere which is the intersection of those planes, the angle between the tangents is the same as the angle contained between the planes. Thus, let two arcs of great circles PA, PB, (fig. 1.) intersect in P, and let two tangents be drawn to them, viz. PD which will be in the plane POA, and PE in the plane POB; then since PD, PE, are both perpendicular to PO, $\angle APB = \angle DPE =$ inclination of the planes in which the arcs are situated.

4. Let PA, PB, be produced to meet the great circle of which P is the pole in M, N, aud any small circle of which P is the pole in A, B; and join OM, ON, CA, CB; then since PD, PE are respectively parallel to OM, ON, $\angle DPE = \angle MON$; and therefore $\angle APB = \angle MON = MN$, employing the arc, according to a preceding remark, to ex-

press the angle which it subtends at the center. This shews that if two arcs containing any angle be produced till each is a quadrant, the arc joining their extremities (which will be a portion of the great circle of which their point of intersection is the pole) will measure the angle they include.

Again, AC the radius of the small circle AB

$$= OA \sin AOP = OA \sin PA\,;$$

and length of arc $AB = AC \times$ circular measure of $\angle ACB$

$$= OA \sin PA \times \text{circular measure of } \angle MON$$

$$= \text{length of arc } MN \times \sin PA.$$

5. The planes of all great circles passing through P will contain OP, and therefore be perpendicular to the plane MON; therefore all great circles passing through P will cut the great circle MN, of which P is the pole, at right angles. Great circles which pass through the pole of another great circle are called *secondaries* to the latter.

Hence, a great circle MN being given, its pole P is determined either (1) by measuring an arc MP equal to a quadrant on any great circle perpendicular to MN; or (2) by drawing any two great circles, not in the same plane, at right angles to MN, and producing them till they intersect in P. And, conversely, if a point on the sphere be such that an arc of a great circle drawn from it perpendicular to a proposed circle is a quadrant, or such that quadrants of two different great circles are intercepted between it and the proposed circle, then that point is the pole of the proposed circle.

6. The arc joining the poles of two great circles subtends an angle at the center equal to their inclination; and the point of intersection of the great circles (i. e. the extremity of the diameter in which their planes intersect) is the pole of the great circle in which their poles lie.

Let P and Q (fig. 2.) be the poles of AC, BC, two great circles whose planes intersect in the diameter OC, then each of the angles POC, QOC, is a right angle, therefore CO is perpendicular to the plane POQ, and consequently C is the pole of the great circle $PQAB$ passing through P

and Q; and since each of the angles POA, QOB, is a right angle,

$$\angle POQ = \angle AOB = \angle ACB.$$

7. The portion of the surface of a sphere contained by three arcs of great circles which cut one another two and two, is called a spherical triangle; the planes in which the arcs lie forming a solid angle at the center of the sphere. The objects of investigation in Spherical Trigonometry are the relations subsisting between the angles at which the three plane faces containing a solid angle are inclined to one another, and the angles which the lines of intersection of those plane faces, or the three edges of the solid angle, form with one another.

Let O (fig. 3.) be the vertex of a solid angle contained by the three plane faces AOB, BOC, COA, and let the arcs of great circles AB, BC, CA be the intersections of these planes with the surface of a sphere described from center O with any radius; then the inclinations of the planes AOB, BOC, COA, to one another are identical with the angles A, B, C, of the spherical triangle; and the three angles at O are proportional to the sides AB, BC, CA; on this account the spherical triangle ABC, which is called the base of the solid angle at O, may be employed with great advantage in conducting the investigation of the relations of the angles of inclination of the faces and edges of the solid angle to one another, which, as has been said, are the proper objects of our research.

The sense in which the spherical triangle is employed being once understood, we may transfer our attention from the solid angle to the triangle in which its faces cut the sphere, and the solid angle need not be represented in our diagrams; but we must still keep in mind that, not the arcs forming the sides of the spherical triangle, but the *angles* which those arcs subtend at the center, are concerned in the calculations, so that the magnitude of the radius of the sphere is of no importance whatever.

8. The three angles of a spherical triangle are usually denoted by A, B, C; the spherical angle A, or the angle BAC contained by the arcs AB, AC, being, as explained above,

the angle between the tangents to those arcs, or between the planes in which the arcs lie; and the angles which the sides BC, CA, AB, respectively opposite to A, B, C, subtend at the center of the sphere, are denoted by a, b, c. When for the sake of brevity, the expression side BC, or side a, is used, the angle subtended at the center of the sphere by the arc BC which is opposite to the spherical angle A, is invariably meant.

9. Since, of the three plane angles which contain a solid angle, any one is less than two right angles, and less than the sum of the two others, and the sum of the three is less than four right angles; therefore, of the three arcs forming the sides of a spherical triangle, which are the measures of those angles, (1) any one is less than the semi-circumference, (2) any one is less than the sum of the two others, and (3) the sum of the three arcs is less than the circumference of a great circle.

Also if the great circle of which one side of a triangle is a portion, be completed, the hemisphere which it bounds will include the triangle; for if not, the points of intersection of two great circles would be separated by an arc greater than the semi-circumference.

It is easily seen that if sides greater than a semi-circumference were admitted, the same angular points might belong to different triangles; as, for instance, to the triangle formed by the arcs BC, CD, and BGD (fig. 5.), as well as to the triangle $BCDH$; and that the solution of such triangles (if they should ever present themselves) can be made to depend on that of triangles limited to have no side greater than a semi-circumference.

10. If a spherical triangle be described on a sphere with the angular points of a given triangle for the poles of its sides, then the angular points of the triangle so described will be the poles of the sides of the given triangle; and its sides and angles will be respectively the supplements of the opposite angles and sides of the given triangle.

Let ABC (fig. 4.) be a spherical triangle, $A'B'C'$ another spherical triangle whose sides have the angular points of the

former for their poles. Join $A'C$, $A'B$, by arcs of great circles, and produce AB, AC, on the surface of the sphere to meet $B'C'$ in G and H; then CA' is a quadrant, because C is the pole of $A'B'$, and BA' is a quadrant because B is the pole of $A'C'$; therefore (Art. 5.) the great circle of which A' is the pole passes through B and C, or A' is the pole of BC; and similarly B' and C' may be shewn to be the poles of AC and AB. From this property, $A'B'C'$ is called with respect to ABC the *Polar* triangle. By construction ABC is the polar triangle of $A'B'C'$.

Also (Art. 4.) $\angle A = HG = C'G - C'H = C'G + B'H - B'C'$

$= 180^0 - B'C'$, since $C'G$, $B'H$ are each quadrants; that is, the angle A is the supplement of the angle subtended by $B'C'$ the side of the Polar triangle of which A is the pole, or which is opposite to A. Similarly it may be proved that $\angle B$, and $\angle C$, are the supplements of the angles subtended by $A'C'$, $A'B'$, the sides of the Polar triangle of which B and C are the poles. From this property the triangle $A'B'C'$ is sometimes called the *supplemental* triangle of ABC.

Hence between the angles of the given triangle, and the sides a', b', c', of the supplemental triangle, we have the relations

$$A + a' = 180^0, \quad B + b' = 180^0, \quad C + c' = 180^0.$$

Also since ABC is the supplemental triangle of $A'B'C'$,

$$\angle A' = 180^0 - BC, \quad \angle B' = 180^0 - AC, \quad \angle C' = 180^0 - AB;$$

so that between the sides of the given triangle and the angles of the supplemental triangle we have the relations,

$$A' + a = 180^0, \quad B' + b = 180^0, \quad C' + c = 180^0.$$

11. This shews that any solid angle contained by three planes being given, if through its vertex three planes be drawn perpendicular to its edges, these will form another solid angle whose edges are perpendicular to the faces of the former; and the inclinations of the faces and edges of the latter will be supplementary to the inclinations of the edges and faces, respectively, of the former.

12. The above Proposition is of great importance; for if any relation be proved to subsist among the angles and sides of a spherical triangle A, B, C, a, b, c, it will also hold for A', B', C', a', b', c' the angles and sides of the supplemental triangle, that is, for $180^\circ - a$, $180^\circ - b$, &c.; hence any formula involving the sides and angles of a spherical triangle will still be true, when throughout it for the angles the supplements of the sides are substituted, and for the sides the supplements of the angles.

13. Since if a', b', c', be the sides of the supplemental triangle,

$$A + B + C + a' + b' + c' = 6 \text{ right angles},$$

and $a' + b' + c'$ always lies between 0 and 4 right angles, therefore $A + B + C$ always lies between six right angles and two right angles. A spherical triangle, therefore, different from a plane triangle, has not the sum of its angles invariable; nor can it have more than two of them $< 60^0$; but it may have two or all of them obtuse, or two or all of them right angles; in which latter case it will include one eighth of the surface of the sphere.

14. The area of a spherical triangle is the same fraction of the area of a hemisphere, that the excess of the sum of its three angles above two right angles is of 360°.

Let ABC (fig. 5.) be a spherical triangle; produce the arcs which contain its angles till they meet again two and two, which will happen when each has become equal to the semi-circumference. The triangle is now common to three different *lunes* (or portions of the spherical surface contained by two semi-circumferences) viz. $ABHDC$, $BCEGA$, and $CBFA$, the latter of which is equal to the sum of the triangles ABC and DCE, for DCE and ABF are evidently equal to one another, since they form the bases of vertically-opposite solid angles at O. Now the area of a lune is the same fraction of the area of the hemisphere (S), that the angle between the two semi-circumferences which contain it, is of 180^0; hence, by equating the two values of the area of each of the above mentioned lunes, we have

$$\Delta\ ABC + BHDC = \frac{A}{180^0} S,$$

$$\text{lune } BCEGA = \frac{B}{180^0} S,$$

$$\Delta\ ABC + DCE = \frac{C}{180^0} S,$$

therefore by addition we get

$$2\,\Delta ABC + S = \frac{A+B+C}{180^0} \,.\, S,$$

$$\text{or, area of triangle } ABC = \frac{A+B+C-180^0}{360^0} \,.\, S.$$

The excess of the three angles above two right angles is called the *spherical excess;* the expression for the area cannot be negative, nor $> S$. (Art. 13.)

Relations between the sides and angles of a spherical triangle.

15. A solid angle such as has been before alluded to, has six elements, namely, the inclinations of the three plane faces to one another, and the inclinations of the three edges to one another; and when any three of these are given, the solid angle is, in general, completely determined. Similarly, a spherical triangle has six elements, namely, the three sides a, b, c, and the three angles A, B, C, respectively opposite to them; and it is in general completely determined when any three of these are given, as there exist relations between the three given elements and each of the unknown ones, by means of which the values of the unknown ones can be obtained. Hence every formula of solution will involve four of the elements, and there can be only four distinct combinations; the first of three sides and an angle; the second of two sides and two angles respectively opposite to them; the third of two sides and two angles one of which is included by the sides; and the fourth of a side and three angles. We shall now investigate the formula belonging to each of these cases.

16. To find a relation between the three sides and any angle of a spherical triangle.

Let the tangent at A (fig. 3.) to the arc AB meet OB produced (which is in the same plane with it) in D, and the tangent to the arc AC meet OC produced in E, and join DE. Then, equating the two values of the square of DE, obtained from the triangles DAE, DOE, we get

$$AD^2 + AE^2 - 2\,AD.AE\cos DAE = OD^2 + OE^2 - 2\,OD.OE\cos DOE;$$

therefore, observing that $OD^2 - AD^2 = OA^2$, $OE^2 - AE^2 = OA^2$, since the angles OAD, OAE, are right angles, and that $\angle\, DOE = BC = a$, $\angle\, DAE = BAC = A$, we find

$$2\,OD\,.\,OE\cos DOE = 2\,OA^2 + 2\,AD\,.\,AE\cos DAE;$$

$$\therefore \cos DOE = \frac{OA}{OE}\cdot\frac{OA}{OD} + \frac{AE}{OE}\cdot\frac{AD}{OD}\cos DAE,$$

$$\text{or } \cos a = \cos b\cos c + \sin b\sin c\cos A,$$

which is the fundamental formula of Spherical Trigonometry.

17. The construction from which this result is obtained, fails unless the two sides b and c that contain the angle A are both less than 90^0, but puts no limitations upon the values of A and a. For the formula to be generally applicable to any angle of any spherical triangle whatever, the proof must be extended to the cases where b and c are, one or both, greater than 90^0, or, one or both, equal to 90^0.

First, suppose that only one of the containing sides b is greater than 90^0, and produce the sides CA, CB, (fig. 6.) to intersect again in C'; then since AB, AC', are both less than 90^0, the formula is applicable to the angle $BAC' = 180^0 - A$ of the triangle BAC';

$$\therefore \cos(180^0 - a) = \cos(180^0 - b)\cos c$$
$$+ \sin(180^0 - b)\sin c\cos(180^0 - A),$$
$$\text{or } \cos a = \cos b\cos c + \sin b\sin c\cos A,$$

the same result as would have been obtained if the formula had been applied immediately to the angle A of the triangle ABC.

Next, suppose that both the containing sides b and c are greater than 90^0, and produce them to intersect again in A' (fig. 7.); then the formula is applicable to the angle $A' = A$ of the triangle $BA'C$;

$$\therefore \cos a = \cos(180^0 - b)\cos(180^0 - c) + \sin(180^0 - b)\sin(180^0 - c)\cos A,$$

$$\text{or } \cos a = \cos b \cos c + \sin b \sin c \cos A,$$

which shews that the formula holds for an angle contained by two sides both greater than 90^0.

Again, suppose $AC = b = 90^0$ (fig. 7.), and describe the arc DC from the point A as pole; then, provided $DC = A$ be different from 90^0, the formula is applicable to the angle D of the triangle BDC, and gives

$$\cos a = \cos(90^0 \sim c)\cos A + \sin(90^0 \sim c)\sin A \cos 90^0,$$

$$\text{or } \cos a = \sin c \cos A,$$

the same result as if the formula had been applied directly to the triangle ABC. If, in this case, DC is 90^0, then BC also is 90^0 (Art. 5), and the triangle is one with two quadrantal sides, and, consequently, with two right angles, for either of which the formula is evidently verified, as it leads to the equation $0 = 0$. If both the containing sides b and c be 90^0, then $a = A$, which agrees with the result given by the formula, namely $\cos a = \cos A$.

18. If $a = b$, then $\cos A = \dfrac{\cos a\,(1 - \cos c)}{\sin a \sin c} = \cos B$;

$\therefore A = B$, since neither of them can exceed 180^0. Conversely, from the Polar Triangle it follows that if the angles at the base are equal, the sides opposite to them are equal.

Also if $A > B$, making $\angle DAB = \angle DBA$ (fig. 8.), we have $DA = DB$, and consequently $CD + DB = CD + DA > AC$, or $a > b$; that is, the greater side is opposite to the greater angle.

19. If the formula which we have just proved to express the relation between the three sides and any angle of any triangle, be applied in succession to the three angles A, B, C, it gives

$$\cos a = \cos b \cos c + \sin b \sin c \cos A \qquad (1),$$

$$\cos b = \cos a \cos c + \sin a \sin c \cos B,$$

$$\cos c = \cos a \cos b + \sin a \sin b \cos C,$$

three equations which comprehend implicitly the whole of spherical trigonometry; since when any three elements are given, they enable us to find all the rest. As however it is desirable that each of the unknown elements should enter singly into the formula by which it is determined, we proceed to investigate the other three distinct formulæ which, in conjunction with (1), will always effect that object.

20. To find a relation between two sides and two angles respectively opposite to them.

We have $\cos A = \dfrac{\cos a - \cos b \cos c}{\sin b \sin c}$;

$$\therefore \sin^2 A = 1 - \cos^2 A = 1 - \frac{(\cos a - \cos b \cos c)^2}{\sin^2 b \sin^2 c}$$

$$= \frac{(1 - \cos^2 b)(1 - \cos^2 c) - (\cos a - \cos b \cos c)^2}{\sin^2 b \sin^2 c}$$

$$= \frac{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c}{\sin^2 b \sin^2 c};$$

$$\therefore \frac{\sin A}{\sin a} = \frac{\sqrt{(1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c)}}{\sin a \sin b \sin c},$$

taking the radical with a positive sign because $\sin A$ and $\sin a$ are both positive; now the second member is a symmetrical function of a, b, c, that is, an expression whose value remains unaltered when a, b, c are interchanged in any manner; consequently the ratio $\sin A \div \sin a$, has a constant value for each of the angles of the triangle; hence we have the three equations

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c},$$

which shew, since this result is as general as the one from which it is deduced, that in any spherical triangle what-

ever, the sines of the angles are to one another as the sines of the sides opposite to them.

The direct method of proving this would be to obtain values of $\sin c$ and $\cos c$ from the two former of the equations of Art. 19, and substitute them in the formula $\sin^2 c + \cos^2 c = 1$; but the above is a simpler process.

21. The preceding proposition may be proved independently as follows.

Drop the perpendicular AP (fig. 3.) upon the plane face BOC, and the perpendiculars PM, PN, upon the edges OB, OC, of the solid angle whose vertex is O and base the spherical triangle ABC, and join AM, AN. Then a line through P parallel to OM would be at right angles to the lines AP, PM, and therefore to the plane APM; consequently OM is perpendicular to the plane AMP, and therefore $\angle AMP = \angle B$; similarly $\angle ANP = \angle C$.

Hence $AM \sin B = AP = AN \sin C$;

$$\text{and } AM = AO \sin AB, \quad AN = AO \sin AC,$$

$$\therefore \sin AB \,.\, \sin B = \sin AC \,.\, \sin C.$$

In the figure we have supposed the two sides AB, AC, to be both less than $90°$. Suppose only one of them AC (fig. 6.) greater than $90°$, then from the triangle $AC'B$,

$$\sin c \sin (180° - B) = \sin (180° - b) \sin C',$$

$$\text{or } \sin c \sin B = \sin b \sin C.$$

If both AB and AC be greater than $90°$, then from the triangle $A'BC$ (fig. 7.)

$$\sin (180° - c) \sin (180° - B) = \sin (180° - b) \sin (180° - C),$$

$$\text{or } \sin c \sin B = \sin b \sin C.$$

If $AC = 90°$, then from the triangle ADB, (fig. 6.) C being the pole of AD, (unless $AB = 90°$) $\sin AD \sin D = \sin AB \sin B$, or $\sin C = \sin c \sin B$. If AB also $= 90°$, the triangle has two quadrantal sides, and consequently two right angles opposite to them, for which the formula is evidently verified. These results prove the relation above stated, to subsist for the angles, and sides opposite to them, of any triangle whatever.

22. To find a relation between two sides, and two angles one of which is included by the sides; for instance between a, b, A, C.

In the equation $\cos a = \cos b \cos c + \sin b \sin c \cos A$

substituting for $\cos c$ and $\sin c$ their values

$$\cos c = \cos a \cos b + \sin a \sin b \cos C$$

$$\sin c = \sin a \frac{\sin C}{\sin A},$$

we find

$$\cos a = \cos a \cos^2 b + \sin a \sin b \cos b \cos C + \sin b \sin a \frac{\sin C}{\sin A} \cos A;$$

therefore, transposing $\cos a \cos^2 b$, and observing that

$$\cos a - \cos a \cos^2 b = \cos a \sin^2 b,$$

and dividing the whole by $\sin a \sin b$, we find

$$\cot a \sin b = \cot A \sin C + \cos b \cos C.$$

This formula is as general as those from the combination of which it has been deduced; and the remembering of it may be facilitated by observing that each member begins with a cotangent multiplied by a sine, the first with any two sides taken at random, the second with two angles of which the former only is opposite to the former of the sides already involved; the last term is the product of the cosines of the elements whose sines are already involved.

23. To find a relation between the three angles and any side of a spherical triangle.

The simplest mode of effecting this is to apply the fundamental formula (1) to the supplemental triangle, which gives

$$\cos a' = \cos b' \cos c' + \sin b' \sin c' \cos A';$$

then, substituting for a', b', c', A', their values in terms of the sides and angles of the proposed triangle, namely,

$$a' = 180^\circ - A, \quad b' = 180^\circ - B, \quad c' = 180^\circ - C, \quad A' = 180^\circ - a,$$

we find for the side a the formula

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a,$$

from which the corresponding formulæ for the sides b and c may be immediately deduced. The direct method of proof in this case would be to eliminate b and c from the three equations of Art. 19.

24. To prove Napier's Analogies.

We have by the preceding Article

$$\cos A + \cos B \cos C = \sin B \sin C \cos a,$$

$$\cos B + \cos A \cos C = \sin A \sin C \cos b,$$

$$\therefore \frac{\cos B + \cos A \cos C}{\cos A + \cos B \cos C} = \frac{\sin A \cos b}{\sin B \cos a} = \frac{\sin a \cos b}{\sin b \cos a};$$

therefore, comparing the difference of the terms of each ratio with the sum of the same terms,

$$\frac{\cos B - \cos A}{\cos B + \cos A} \cdot \frac{1 - \cos C}{1 + \cos C} = \frac{\sin (a - b)}{\sin (a + b)},$$

or $\tan \frac{1}{2}(A + B) \tan \frac{1}{2}(A - B) \tan^2 \frac{1}{2} C$

$$= \frac{\sin \frac{1}{2}(a - b) \cos \frac{1}{2}(a - b)}{\sin \frac{1}{2}(a + b) \cos \frac{1}{2}(a + b)} \text{ (1)}.$$

But $\dfrac{\sin A}{\sin B} = \dfrac{\sin a}{\sin b}$ gives $\dfrac{\tan \frac{1}{2}(A + B)}{\tan \frac{1}{2}(A - B)} = \dfrac{\tan \frac{1}{2}(a + b)}{\tan \frac{1}{2}(a - b)}$.

Multiplying these equations together, and dividing one by the other, and extracting the roots, observing that $\frac{1}{2}(A - B)$ and $\frac{1}{2}(a - b)$ are less than 90^0 and of the same sign, and consequently by equation (1) $\tan \frac{1}{2}(A + B)$ and $\cos \frac{1}{2}(a + b)$ must have the same sign, (and therefore $\frac{1}{2}(A + B)$, $\frac{1}{2}(a + b)$, since neither can exceed 180^0, are both less or both greater than 90^0,) we find

$$\tan \frac{1}{2}(A + B) = \frac{\cos \frac{1}{2}(a - b)}{\cos \frac{1}{2}(a + b)} \cot \frac{1}{2} C,$$

$$\tan \frac{1}{2}(A - B) = \frac{\sin \frac{1}{2}(a - b)}{\sin \frac{1}{2}(a + b)} \cot \frac{1}{2} C.$$

Also, applying these formulæ to the polar triangle, or, which is the same thing, replacing A, B, C, a, b, by $180^0 - a$, $180^0 - b$, $180^0 - c$, $180^0 - A$, $180^0 - B$, we find

$$\tan \tfrac{1}{2}(a+b) = \frac{\cos \frac{1}{2}(A-B)}{\cos \frac{1}{2}(A+B)} \tan \tfrac{1}{2} c,$$

$$\tan \tfrac{1}{2}(a-b) = \frac{\sin \frac{1}{2}(A-B)}{\sin \frac{1}{2}(A+B)} \tan \tfrac{1}{2} c.$$

25. These formulæ, being under the form of analogies or proportions, are known under the name of Napier's Analogies; they are useful in the solution of triangles, the two former when two sides and the included angle are given, the two latter when one side and the two angles adjacent to it are given.

They may be likewise put into the following shape.

Since

$$1 + \cos c = 1 + \cos a \cos b + \sin a \sin b \,(\cos^2 \tfrac{1}{2} C - \sin^2 \tfrac{1}{2} C)$$

$$= \{1 + \cos(a-b)\} \cos^2 \tfrac{1}{2} C + \{1 + \cos(a+b)\} \sin^2 \tfrac{1}{2} C;$$

$$\therefore\ \cos^2 \tfrac{1}{2} c = \cos^2 \tfrac{1}{2}(a-b) \cos^2 \tfrac{1}{2} C + \cos^2 \tfrac{1}{2}(a+b) \sin^2 \tfrac{1}{2} C.$$

Similarly,

$$\sin^2 \tfrac{1}{2} c = \sin^2 \tfrac{1}{2}(a-b) \cos^2 \tfrac{1}{2} C + \sin^2 \tfrac{1}{2}(a+b) \sin^2 \tfrac{1}{2} C.$$

Now add unity to the square of each member of the two first analogies, and we find

$$\sec^2 \tfrac{1}{2}(A+B) = \frac{\cos^2 \frac{1}{2} c}{\cos^2 \frac{1}{2}(a+b) \sin^2 \frac{1}{2} C},$$

$$\sec^2 \tfrac{1}{2}(A-B) = \frac{\sin^2 \frac{1}{2} c}{\sin^2 \frac{1}{2}(a+b) \sin^2 \frac{1}{2} C};$$

or, extracting the roots, since $\frac{1}{2}(A+B)$, $\frac{1}{2}(a+b)$ are both greater or both less than $90°$,

$$\cos \tfrac{1}{2}(A+B) \cos \tfrac{1}{2} c = \cos \tfrac{1}{2}(a+b) \sin \tfrac{1}{2} C,$$

$$\cos \tfrac{1}{2}(A-B) \sin \tfrac{1}{2} c = \sin \tfrac{1}{2}(a+b) \sin \tfrac{1}{2} C;$$

and multiplying these respectively by the two first analogies,

$$\sin \tfrac{1}{2}(A+B) \cos \tfrac{1}{2} c = \cos \tfrac{1}{2}(a-b) \cos \tfrac{1}{2} C,$$

$$\sin \tfrac{1}{2}(A-B) \sin \tfrac{1}{2} c = \sin \tfrac{1}{2}(a-b) \cos \tfrac{1}{2} C.*$$

* These four formulæ are sometimes proved by substituting in the developments of $\cos \frac{1}{2}(A+B)$, $\cos \frac{1}{2}(A-B)$, &c., the values of $\cos \frac{1}{2} A$, $\sin \frac{1}{2} A$, &c., in terms of the sides, found below, Art. 34.

Solution of right-angled triangles.

26. When in the triangle ABC (fig. 10.) the angle A is 90^0, that is, when the plane of the great circle AB is perpendicular to the plane of the great circle AC, the spherical triangle is called right-angled, and the side $BC = a$ is called the hypothenuse. By a right-angled triangle is usually understood a triangle with only one right angle, the two remaining angles being either acute or obtuse; for if there should be two right angles A and B, then the sides opposite to them BC and AC would be quadrants, and the angle C would equal AB; or if there should be three right angles, then all the sides would be quadrants (Art. 4.); so that it is needless to consider these particular cases. As a triangle is in general determined when any three parts are given, a right-angled triangle, with one exception, is determined when any two parts are given. To obtain the formulæ necessary for the solution of right-angled triangles, we have only to make $A = 90^0$ in the fundamental relations which have been investigated for any triangle whatever. These formulæ, by means of which when any two parts of a right-angled triangle are given, the remaining three become known explicitly in terms of the given quantities, are embodied with equal elegance and convenience in Napier's Rules, the enunciation and proof of which are as follows.

27. The right angle being left out of consideration, the two sides including the right angle, and the complements of the hypothenuse and of the two other angles, are called *circular parts*. Any one of these being taken as the *middle* part, the two circular parts which, or their complements, are immediately contiguous to it, or its complement, in position, are called the *adjacent* parts; and the other two parts are called the *opposite* parts. Then, whatever the middle part be, whether a side, or the complement of the hypothenuse, or of an angle,

sine of the middle part

= product of the tangents of the adjacent parts;

sine of the middle part

= product of the cosines of the opposite parts.

1st. Let the complement of the hypothenuse $90^0 - a$ be middle part, then $90^0 - B$, $90^0 - C$, are the adjacent parts, and b, c, the opposite parts; and the formulæ,

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a,$$

$$\cos a = \cos b \cos c + \sin b \sin c \cos A,$$

since $A = 90^0$, and therefore $\cos A = 0$, $\sin A = 1$, become

$\cos a = \cot B \cot C$, or $\sin(90^0 - a) = \tan(90^0 - B)\tan(90^0 - C)$,

$\cos a = \cos b \cos c$, or $\sin(90^0 - a) = \cos b \cos c$.

2nd. Let the complement of an angle $90^0 - C$ be middle part, then $90^0 - a$ and b are the adjacent parts, and $90^0 - B$ and c the opposite parts; and the formulæ

$$\cot a \sin b = \cot A \sin C + \cos C \cos b,$$

$$\cos C = -\cos A \cos B + \sin A \sin B \cos c,$$

become $\cos C = \cot a \tan b$, or $\sin(90^0 - C) = \tan(90^0 - a)\tan b$,

$\cos C = \sin B \cos c$, or $\sin(90^0 - C) = \cos(90^0 - B)\cos c$.

If $90^0 - B$ be taken for middle part, the rules may be proved exactly in the same way.

3rd. Let either side b be middle part, then $90^0 - C$ and c are the adjacent parts, and $90^0 - a$ and $90^0 - B$ the opposite parts, and the formulæ

$$\cot c \sin b = \cot C \sin A + \cos A \cos b$$

$$\sin A \sin b = \sin a \sin B,$$

become $\sin b = \cot C \tan c$, or $\sin b = \tan(90^0 - C)\tan c$,

$\sin b = \sin a \sin B$, or $\sin b = \cos(90^0 - a)\cos(90^0 - B)$.

If c be taken for middle part, the rules may be proved exactly in the same way.

28. We are thus furnished with ten relations amongst the five parts of a right-angled triangle, each being a different combination of three of the parts; but five parts, taken three at a time, can be combined only in 10 different ways; consequently the above Rules, when any two parts whatever are

given, will supply us with formulæ in which each of the remaining parts is separately combined with those two given parts, and in a form adapted for the immediate application of logarithms.

There can evidently be only six distinct cases, viz. 1 and 2 when the hypothenuse is given together with an angle, or with one of the sides containing the right angle; 3 and 4 when one of the sides containing the right angle is given together with the angle adjacent to it, or with the angle opposite to it; 5 and 6 when the two sides containing the right angle are given, or when the two angles are given. In applying Napier's Rules to obtain the three unknown parts from two given ones, it is sometimes requisite to take for middle part one of the given parts, and sometimes one of those that are sought, the sole object being to separately combine each of the unknown parts with the given ones. We shall now go through the six cases of solving right-angled triangles, first however premising the two following observations.

29. The formula $\cos a = \cos b \cos c$ requires that either all three, or only one of the cosines should be positive; therefore the three sides of a right-angled triangle are either all less than 90^0, or two of them greater than 90^0, and the third less.

The formula $\sin c = \cot B \tan b = \tan b \div \tan B$ shews that $\tan B$ and $\tan b$ have the same sign, since $\sin c$ is always positive, c being less than 180^0; therefore B and b, since neither of them can exceed 180^0, are both greater or both less than 90^0; that is, in right-angled triangles a side and the angle opposite to it are either both greater or both less than a right angle; this is usually expressed by saying that they are of the same affection.

30. Hence (fig. 14) if DCD' be a great circle perpendicular to DAD', and CD be less than 90^0, CD is the least and CD' the greatest arc that can be intercepted between the point C and DAD'; and of the arcs so intercepted the nearer any one is to CD, the less it is. For since CD is less than 90^0, $\angle CBD < 90^0 < CDB$, and therefore $CD < CB$. Also $\angle CBD' > 90^0 > CD'B$, and therefore $CD' > CB$. And from the equations $\cos AC = \cos AD \cos CD$, $\cos BC = \cos BD \cos CD$, it is easily seen that $BC < AC$.

31. Case I. Having given the hypothenuse a, and an angle B, to find b, c, C.

Taking successively b, $90^\circ - B$, and $90^\circ - a$, for middle part, we get

$$\sin b = \sin a \sin B, \quad \cos B = \cot a \tan c, \quad \cos a = \cot B \cot C;$$

C and c are determined without ambiguity, and b must be of the same affection as B.

Case II. Having given the hypothenuse a, and a side b, to find c, B, C.

Taking successively $90^\circ - a$, b, $90^\circ - C$ for middle part, we get

$$\cos a = \cos b \cos c, \quad \sin b = \sin a \sin B, \quad \cos C = \cot a \tan b.$$

By these formulæ c and C are determined without ambiguity, for there is only one angle less than 180° corresponding to a given cosine; also B, though determined by its sine, is not ambiguous since it must be of the same affection as b.

Case III. Having given one of the sides containing the right angle b, and the angle adjacent to it C, to find a, c, B.

Taking successively $90^\circ - C$, b, $90^\circ - B$ for middle part, we get

$$\cos C = \tan b \cot a, \quad \sin b = \tan c \cot C, \quad \cos B = \cos b \sin C,$$

which determine a, c, B without ambiguity.

Case IV. Having given one of the sides containing the right angle b, and the angle opposite to it B, to find a, c, C.

Taking successively b, c, $90^\circ - B$, for middle part, we get

$$\sin b = \sin B \sin a, \quad \sin c = \cot B \tan b, \quad \cos B = \sin C \cos b.$$

Here there is an ambiguity, all the unknown parts being determined by their sines; and it is easily seen that such ought to be the case. For if we produce the two sides BA, BC (fig. 11.) till they intersect again in B', we have a second right-angled triangle CAB' in which the side b and the angle $B' = B$ are evidently the same as in the triangle ABC; and the other parts of the second triangle are the supplements of the corresponding parts C, a, c of the first triangle. Hence we may take a either less or greater than 90°, but having made the choice, c will be given by the relation $\cos a = \cos b \cos c$,

and then C will be of the same affection as c; and thus the two triangles, which equally solve the problem, will be determined.

There will however be only one triangle if $b = B$, having two right angles; and none at all, if $\sin b > \sin B$, for then the value of $\sin a$ is impossible. In the latter case, since B and b are of the same affection, b is greater or less than B, according as B is acute or obtuse; and therefore B and b cannot form parts of a right-angled spherical triangle; for if P be the pole of AB, and B of DE, then when B is acute so that ABC is the triangle, $PC < PE$ (Art. 30), and therefore $AC < DE$, i.e. b cannot be greater than B; and when B is obtuse so that ABC' is the triangle, then $PC' > PE'$ and therefore $AC' > DE'$, i.e. b cannot be less than B. Hence when $\angle B$ is acute, there is no triangle if $b > B$, one if $b = B$, and two if $b < B$; and when $\angle B$ is obtuse, there is no triangle if $b < B$, one if $b = B$, and two if $b > B$.

Case V. Having given the two sides containing the right angle b and c, to find a, B, C.

Taking successively $90^\circ - a$, c, b, for middle part, we get

$$\cos a = \cos b \cos c, \quad \sin c = \cot B \tan b, \quad \sin b = \cot C \tan c,$$

which determine a, B, C, without ambiguity.

Case VI. Having given the two oblique angles B and C, to find a, b, c.

Taking successively $90^\circ - a$, $90^\circ - B$, $90^\circ - C$ for middle part, we get

$$\cos a = \cot B \cot C, \quad \cos B = \cos b \sin C, \quad \cos C = \cos c \sin B.$$

These formulæ leave no ambiguity; and if the triangle is impossible they will shew it.

Solution of quadrantal and isosceles triangles.

32. Of other triangles which may be solved as right-angled triangles, the principal class is that called *quadrantal* triangles, in which one side, a, is a quadrant. Since the polar triangle in this case will have one angle $A' = 180^\circ - a = 90^\circ$, applying Napier's Rules to it we get

$$\cos a' = \cot B' \cot C' \quad \text{or} \quad = \cos b' \cos c',$$
$$\cos C' = \cot a' \tan b' \quad \text{or} \quad = \sin B' \cos c',$$
$$\sin c' = \cot B' \tan b' \quad \text{or} \quad = \sin a' \sin C'.$$

Therefore, substituting for a', b', &c. their values, we get

$$-\cos A = \cot b \cot c \quad \text{or} \quad = \cos B \cos C,$$
$$-\cos c = \cot A \tan B \quad \text{or} \quad = -\sin b \cos C,$$
$$\sin C = \cot b \tan B \quad \text{or} \quad = \sin A \sin c;$$

or, $\sin(A-90^0) = \tan(90^0-b)\tan(90^0-c)$ or $= \cos B \cos C$,

$\sin(90^0-c) = \tan(A-90^0)\tan B$ or $= \cos(90^0-b)\cos C$,

$\sin C = \tan(90^0-b)\tan B$ or $= \cos(A-90^0)\cos(90^0-c)$;

which shew that if the complements of the two sides, the complement of the hypothenusal angle taken negatively, and the two other angles be taken for the circular parts, that is, $90^0 - b$, $90^0 - c$, $-(90^0 - A)$, B and C, the quadrantal triangle may be at once solved by Napier's Rules.

33. If a triangle be isosceles, joining the vertex with the middle of the base by the arc of a great circle, we divide it into two right-angled triangles equal in all respects; if therefore any two parts of an isosceles triangle be given, (counting however the two equal sides as only one element and the two equal angles opposite to them only as one element,) all the parts of the triangle may be determined by the Rules for right-angled triangles.

Also, if in a spherical triangle the sum of any two sides $a + b = 180^0$, producing b and c till they intersect again in A' (fig. 7.) we have $b + CA' = 180^0$; $\therefore CB = A'C$; hence the solution of the triangle ABC is reduced to that of the isosceles triangle $A'CB$. The condition $a + b = 180^0$ is evidently the same as $A+B=180^0$; for since $A'C=CB$, $\angle CA'B=CBA'=A$, therefore $A + B = CBA' + CBA = 180^0$.

Solution of oblique-angled triangles.

34. Oblique-angled triangles, in which there must always be given three of the six elements A, B, C, a, b, c, present six distinct cases, the data in them being as follows: 1. three sides; 2. two sides and an angle opposite to one of them;

3. two sides and the included angle; 4. two angles and the side opposite to one of them; 5. two angles and the side adjacent to them both; 6. three angles. All these cases are readily solved by means of the four fundamental relations expressed by the formulæ of Arts. 19—23; but as these formulæ require certain modifications to make them suitable for actual computation by means of logarithms, it is necessary to go through each case separately.

35. Case I. Having given the three sides, to find the angles.

We have (Art. 16.) $\cos A = \dfrac{\cos a - \cos b \cos c}{\sin b \sin c}$,

which gives A, the second member being entirely known; and by similar formulæ may B and C be determined; but this formula is not suited to logarithmic calculation, a defect in it that may be supplied as follows. First we have

$$2 \sin^2 \tfrac{1}{2} A = 1 - \cos A,$$

and substituting for $\cos A$ its value, we get successively

$$2 \sin^2 \tfrac{1}{2} A = 1 - \frac{\cos a - \cos b \cos c}{\sin b \sin c} = \frac{\cos b \cos c + \sin b \sin c - \cos a}{\sin b \sin c}$$

$$= \frac{\cos (b - c) - \cos a}{\sin b \sin c} = \frac{2 \sin \frac{1}{2} (a + b - c) \sin \frac{1}{2} (a - b + c)}{\sin b \sin c}.$$

To simplify this, let half the perimeter of the triangle be denoted by s, so that $a + b + c = 2s$; $\therefore a + b - c = 2(s - c)$, $a - b + c = 2(s - b)$; hence substituting, and extracting the root, we find

$$\sin \tfrac{1}{2} A = \sqrt{\frac{\sin (s - b) \sin (s - c)}{\sin b \sin c}}.$$

With equal facility may similar expressions for $\cos \frac{1}{2} A$ and $\tan \frac{1}{2} A$ be found. For

$$2 \cos^2 \tfrac{1}{2} A = 1 + \cos A = 1 + \frac{\cos a - \cos b \cos c}{\sin b \sin c} = \frac{\cos a - \cos (b + c)}{\sin b \sin c}$$

$$= \frac{2 \sin \frac{1}{2} (a + b + c) \sin \frac{1}{2} (b + c - a)}{\sin b \sin c} = \frac{2 \sin s \sin (s - a)}{\sin b \sin c},$$

$$\therefore \cos \tfrac{1}{2} A = \sqrt{\frac{\sin s \sin (s-a)}{\sin b \sin c}}.$$

Also $\tan \frac{1}{2} A = \sin \frac{1}{2} A \div \cos \frac{1}{2} A = \sqrt{\dfrac{\sin (s-b) \sin (s-c)}{\sin s \sin (s-a)}}$.

These formulæ determine the angle without ambiguity; for since A is the angle of a spherical triangle, $\frac{1}{2} A$ is less than 90^0. The principles which must guide us in selecting a formula for any particular case, are the same as those explained in Plane Trigonometry, Art. 113.

36. Taking twice the product of the values of $\sin \frac{1}{2} A$ and $\cos \frac{1}{2} A$, we find

$$\sin A = \frac{2}{\sin b \sin c} \sqrt{\sin s \sin (s-a) \sin (s-b) \sin (s-c)};$$

which, as it requires seven logarithms, is not an advantageous formula for the calculation of A. It is the form to which the expression for $\sin A$ in Art. 20 may be reduced.

37. Case II. Having given two sides a, b, and the angle A opposite to one of them, to find c, B, C.

The angle B opposite to the side b may first of all be obtained from the relation

$$\sin B = \frac{\sin A \sin b}{\sin a}.$$

Then C and c may be determined by Napier's Analogies, which give

$$\tan \tfrac{1}{2} C = \frac{\cos \frac{1}{2} (a-b)}{\cos \frac{1}{2} (a+b)} \cot \tfrac{1}{2} (A+B),$$

$$\tan \tfrac{1}{2} c = \frac{\cos \frac{1}{2} (A+B)}{\cos \frac{1}{2} (A-B)} \tan \tfrac{1}{2} (a+b).$$

38. As the angle B is determined by its sine, it may be either greater or less than 90^0; this case, consequently, will often admit of two solutions; but for certain values of the given elements a, b, A there will be only one triangle, or none at all, under the usual restriction of having no side greater

than 180°. To the examination of these circumstances, analogous to what happens in the second case of Plane Triangles (Art. 105.), we shall afterwards recur. The ambiguity to a certain extent may be removed by observing that, as $\frac{1}{2}(a+b)$ and $\frac{1}{2}(A+B)$ are of the same affection (Art. 24.), if $a+b>180^\circ$, then $A+B>180^\circ$; if therefore A be less than 90°, B must be greater than 90°, and consequently there can be only one triangle having the given elements, although there may be none; but if A be greater than 90°, then a value of B either greater or less than 90° will satisfy the condition $A+B>180^\circ$, and consequently there may be two triangles having the given elements, although there may be none. Again if $a+b<180^\circ$, then $A+B<180^\circ$; if therefore A be greater than 90°, B must be less than 90°, and there can be only one solution; but if A be less than 90°, then a value of B either greater or less than 90° will satisfy the condition $A+B<180^\circ$, and there may be two solutions.

39. We may determine C and c directly from the given quantities, without first finding B. From Art. 22, we get

$$\cot a \sin b = \cot A \sin C + \cos b \cos C = \cos b \left(\cos C + \sin C \frac{\cot A}{\cos b}\right).$$

Let ϕ be a subsidiary angle determined by the equation

$$\tan \phi = \frac{\cot A}{\cos b};$$

$$\therefore \cot a \tan b = \frac{\cos (C-\phi)}{\cos \phi}, \text{ or } \cos (C-\phi) = \frac{\tan b \cos \phi}{\tan a},$$

from which $C-\phi$, and consequently C, may be obtained.

Again, we have $\cos a = \cos b \cos c + \sin b \sin c \cos A$

$$= \cos b (\cos c + \sin c \tan b \cos A).$$

Let θ be a subsidiary angle determined by the equation $\tan \theta = \tan b \cos A$;

$$\therefore \cos a = \cos b \,.\, \frac{\cos (c-\theta)}{\cos \theta}, \text{ or } \cos (c-\theta) = \frac{\cos a \cos \theta}{\cos b},$$

from which $c-\theta$, and consequently c, may be obtained.

40. With respect to such transformations as those just made, and which are of constant occurrence in this subject, it may be observed that their object in every case is to change into a product a binomial of the form $m \sin a + n \cos a$; this is done by first making one of the quantities m or n a factor of the whole expression, so that it becomes $m\left(\sin a + \frac{n}{m}\cos a\right)$, and then equating $\frac{n}{m}$ to the tangent or cotangent of an angle (ϕ), which is always allowable as the tangent and cotangent are susceptible of all values; by this substitution the expression is reduced to

$$\frac{m \sin (a + \phi)}{\cos \phi} \text{ or } \frac{m \cos (a - \phi)}{\sin \phi}.$$

41. It is also worth remarking that the introduction of the subsidiary angles ϕ and θ amounts to dividing the proposed triangle into two right-angled triangles. Thus, in the present case, if $CD = h$ (fig. 13.) be an arc of a great circle drawn from the angle included by the given sides perpendicular to the opposite side, and if $\angle ACD = \phi$ and the segment $AD = \theta$, then from the right-angled triangles ACD, BCD, by Napier's Rules, we get

$$\cos b = \cot \phi \cot A, \quad \cos A = \cot b \tan \theta,$$

$$\cos (C - \phi) = \tan h \cot a = \frac{\cos \phi}{\cot b} \cdot \cot a,$$

$$\cos a = \cos h \cos (c - \theta) = \frac{\cos b}{\cos \theta} \cdot \cos (c - \theta),$$

which are identical with the former results, and involve the same ambiguities.

42. Case III. Having given two sides a, b, and the included angle C, to find A, B, c.

By Napier's Analogies we have

$$\tan \tfrac{1}{2}(A + B) = \frac{\cos \frac{1}{2}(a - b)}{\cos \frac{1}{2}(a + b)} \cot \tfrac{1}{2} C,$$

$$\tan \tfrac{1}{2}(A - B) = \frac{\sin \frac{1}{2}(a - b)}{\sin \frac{1}{2}(a + b)} \cot \tfrac{1}{2} C,$$

which determine $\frac{1}{2}(A+B)$ and $\frac{1}{2}(A-B)$, and consequently A and B; and these being known, we obtain c from the equation, $\sin c = \frac{\sin C}{\sin A} \sin a$, observing that the greater side is opposite to the greater angle. Or, c may be found by one of the formulæ of Art. 25, and then only two additional logarithms will be wanted.

43. If it be required to determine c independently of A and B, we have

$$\cos c = \cos a \cos b + \sin a \sin b \cos C = \cos a\,(\cos b + \sin b \tan a \cos C).$$

Let θ be a subsidiary angle determined by the equation $\tan\theta = \tan a \cos C$;

$$\therefore \cos c = \frac{\cos a \cos (b-\theta)}{\cos\theta}$$

which gives c without ambiguity.

Also A may be independently determined from the formula

$$\cot A = \frac{\cot a \sin b - \cos b \cos C}{\sin C} = \cot C\left(\frac{\cot a}{\cos C}\sin b - \cos b\right),$$

which, introducing the same subsidiary angle θ, becomes

$$\cot A = \frac{\cot C \sin (b-\theta)}{\sin\theta}.$$

It is easily seen that these results may be obtained by dropping the arc BD' perpendicular to AC, and calling the segment $CD' = \theta$. (fig. 13.)

The side c and the angle B may in a similar manner be directly found from the given quantities by dropping a perpendicular from the angle A upon the opposite side.

44. Case IV. Having given two angles A and B, with the side adjacent to both c, to find a, b, C.

By Napier's Analogies we have

$$\tan \tfrac{1}{2}(a+b) = \frac{\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)} \tan\tfrac{1}{2}c,$$

$$\tan \tfrac{1}{2}(a-b) = \frac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)} \tan\tfrac{1}{2}c,$$

which determine $\frac{1}{2}(a+b)$, $\frac{1}{2}(a-b)$, and consequently a and b; and these being known, we have C from the equation

$$\sin C = \frac{\sin c}{\sin a} \sin A,$$

observing that the greater side is opposite the greater angle.

45. If it be required to determine C independently of a and b, we have

$$\cos C = -\cos A \cos B + \sin A \sin B \cos c$$
$$= \cos A\,(-\cos B + \sin B \tan A \cos c),$$

and determining the subsidiary angle ϕ by the equation

$\cot\phi = \tan A \cos c$, we get $\cos C = \dfrac{\cos A \sin (B-\phi)}{\sin\phi}$.

Also a may be independently determined from the formula

$$\cot a \sin c = \cot A \sin B + \cos c \cos B = \cos c \left(\cos B + \sin B \frac{\cot A}{\cos c}\right),$$

which, introducing the same subsidiary angle ϕ, becomes

$$\cot a = \frac{\cot c \cos (B-\phi)}{\cos\phi}.$$

It is easily seen that these results may be obtained by dropping the arc BD' perpendicular to AC and calling $\angle ABD' = \phi$.

The side b and $\angle C$ may in a similar way be determined directly from the given quantities by dropping the perpendicular from the angle A. This case is analogous to the third and gives rise to no ambiguity.

46. Case V. Having given two angles A, B, and the side a opposite to one of them, to find b, c, C.

This case, being exactly analogous to the second, is treated in the same way, and gives rise to the same ambiguities.

The side b may be first of all obtained from the formula

$\sin b = \dfrac{\sin B}{\sin A} \sin a$, and then C and c from the formulæ

$$\tan \tfrac{1}{2} C = \frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)} \cot \tfrac{1}{2}(A+B),$$

$$\tan \tfrac{1}{2} c = \frac{\cos \frac{1}{2}(A+B)}{\cos \frac{1}{2}(A-B)} \tan \tfrac{1}{2}(a+b).$$

47. We may also determine C and c directly from the given quantities; for we have

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a$$

$$= \cos B(-\cos C + \sin C \tan B \cos a) = \frac{\cos B \sin (C-\psi)}{\sin \psi},$$

ψ being a subsidiary angle determined from the equation $\cot \psi = \tan B \cos a$. Also

$$\cot a \sin c = \cot A \sin B + \cos c \cos B,$$

if we determine ϕ by the equation $\cot \phi = \dfrac{\cot a}{\cos B}$, becomes

$$\cot A \sin B = \cos B\left(-\cos c + \sin c \frac{\cot a}{\cos B}\right) = \frac{\cos B \sin (c-\phi)}{\sin \phi}.$$

These results may evidently be obtained by dropping the arc CD perpendicular on AB, and calling $\angle BCD = \psi$, and the segment $BD = \phi$.

48. Case VI. Having given the three angles A, B, C, to find the sides a, b, c.

The formulæ of solution in this case are obtained by a process exactly similar to that in Case I. We have

$$\cos a = \frac{\cos A + \cos B \cos C}{\sin B \sin C};$$

$$\therefore\ 2 \sin^2 \tfrac{1}{2} a = 1 - \frac{\cos A + \cos B \cos C}{\sin B \sin C} = -\frac{\cos A + \cos (B+C)}{\sin B \sin C}$$

$$= -\frac{2 \cos \frac{1}{2}(A+B+C) \cos \frac{1}{2}(B+C-A)}{\sin B \sin C}.$$

Let $A + B + C = 2S$; $\therefore B + C - A = 2(S - A)$; hence substituting, and extracting the root,

$$\sin \tfrac{1}{2} a = \sqrt{\frac{-\cos S \cos (S - A)}{\sin B \sin C}}.$$

Similarly, $$\cos \tfrac{1}{2} a = \sqrt{\frac{\cos (S - B) \cos (S - C)}{\sin B \sin C}},$$

$$\tan \tfrac{1}{2} a = \sqrt{\frac{-\cos S \cos (S - A)}{\cos (S - B) \cos (S - C)}}.$$

These values, which might have been derived from the formulæ of Case I. by means of the Polar Triangle, are always real. In the first place S is always greater than 90^0 and less than 270^0, therefore $\cos S$ is always negative; also because in the supplementary triangle, $a' < b' + c'$,

$$\therefore\ 180^0 - A < 180^0 - B + 180^0 - C, \text{ or } B + C - A < 180^0;$$

consequently $S - A < 90^0$, and its cosine is positive; and in like manner $\cos (S - B)$, $\cos (S - C)$, are positive.

Ambiguous cases in the solution of oblique-angled triangles.

49. It appears that the only cases in which any doubt can arise as to the values of the computed elements, are the second and the fifth; and it becomes necessary to investigate the conditions to which the given elements are respectively subject, when they correspond to two triangles, or to only one, or to none at all.

Let ADD', DCD', (fig. 14.) be two great circles at right angles to one another, and ACE a circle making an acute angle CAD with ADD'; then (Art. 30) CD is less than 90^0 and is the least arc, and CD' is the greatest arc, that can be intercepted between C and the circle ADD'; and of the arcs so intercepted, those which are equally inclined to CD or CD' are equal to one another, and the nearer any arc is to CD the less it is, and the nearer any arc is to CD' the greater it is. Suppose now $\angle CAD = A$, or $\angle CAD' = A$, to be the given angle according as it is acute or obtuse, and $AC = b$ the given adjacent side; then if the given opposite side a be intermediate in magnitude to

the perpendiculars CD and CD', the small circle described with pole C and angular radius $= a$ will always cut ADD' in two points, and determine two admissible triangles having the proposed elements A, b, a; except in the cases in which one or both of the triangles become inadmissible on account of having a side $> 180^0$, or an angle $= 180^0 - A$ instead of A.

But if the side a be less than CD, or greater than CD', the small circle described from C as pole with angular radius $= a$, will never intersect ADD', but fall entirely above it in the former case, and below it in the latter; and the construction of a triangle with the proposed elements will be impossible. This also appears from the formula; for if $a < CD$, then since $CD < 90^0$, $\sin a < \sin CD < \sin A \sin b$; and if $a > CD'$, then since $CD' > 90^0$, $\sin a < \sin CD' < \sin A \sin b$; so that in both cases the equation $\sin B = \sin A \sin b \div \sin a$ for determining B, is impossible.

Excluding therefore these impossible cases, and always supposing the circles to cut one another, we shall now examine in what cases one or both of the triangles determined by their intersections are inadmissible on account of having a side $> 180^0$, or an angle $= 180^0 - A$ instead of A.

50. Case I. Let the given angle A be less than 90^0; and first suppose b less than 90^0, then AD is less than 90^0 (Art. 30.) and less than DE. If therefore a be less than b, it is evident that we may place an arc $BC = a$ between AC and CD, and another $CB' = a$ between EC and CD, and so construct two triangles that have the given elements; but if $a = b$ the triangle ABC disappears, and if $a > b$ the triangle ABC has the angle $180^0 - A$ instead of A, so that there only remains one triangle ACB' with the proposed elements, and that, only so long as $a < 180^0 - b$. Next suppose AC* or $b > 90^0$, then if $a < 180^0 - b$ we may place $CB = CB' = a$ on each side of the perpendicular CD, and so construct two triangles that have the given elements; but if $a = \pi - b$ the triangle ACB' becomes a lune, and if $a > \pi - b$ the triangle ACB' has a side $> 180^0$, so that there only remains one triangle ACB that satisfies the conditions, and that, only so long as $a < b$.

* The reader is requested to supply the letters in this part of the diagram.

Hence we conclude that there are, if

$$b < 90^0 \begin{cases} a < b & \text{two solutions,} \\ a = \text{or} > b & \text{one solution,} \\ a + b = \text{or} > 180^0 & \text{no solution;} \end{cases}$$

$$b > 90^0 \begin{cases} a < 180^0 - b & \text{two solutions,} \\ a = \text{or} > 180^0 - b & \text{one solution,} \\ a = \text{or} > b & \text{no solution.} \end{cases}$$

51. Case II. Let the given angle be greater than 90^0; then in exactly the same way, using the triangle ACD', we may shew that there are, if

$$b < 90^0 \begin{cases} a > 180^0 - b & \text{two solutions,} \\ a = \text{or} < 180^0 - b & \text{one solution,} \\ a = \text{or} < b & \text{no solution;} \end{cases}$$

$$b > 90^0 \begin{cases} a > b & \text{two solutions,} \\ a = \text{or} < b & \text{one solution,} \\ a = \text{or} < 180^0 - b & \text{no solution.} \end{cases}$$

52. It is evident that by means of the Polar Triangle the discussion of the 5th Case of the solution of oblique-angled triangles, in which the data are A, B, a, may be reduced to the above; and we may apply the results just obtained to that case, if we change a, b, A into A, B, a, and the sign $>$ into $<$, and $<$ into $>$. In those cases where the given elements correspond to only one solution, the calculation will still indicate two; but the one to be taken may be always discerned by means of the property that the greater side is opposite the greater angle.

Approximate solution of spherical triangles in certain cases.

53. We shall now give a few instances of the application of Spherical Trigonometry to cases which allow the exact formulæ hitherto obtained, to be advantageously replaced by approximate ones of much greater simplicity. These instances chiefly occur in investigations which have for their object the correct representation of a portion of the Earth's surface which is too large in extent to be considered as situated in one plane.

54. Let α, β, γ be the arcs forming the sides of a spherical triangle situated on the surface of the terrestrial globe whose radius suppose r, and let a, b, c be the circular measures of the angles which those arcs respectively subtend at the Earth's center, then $a = \alpha \div r$, $b = \beta \div r$, $c = \gamma \div r$. Now although the arcs α, β, γ may be several miles in length they are so small compared with the Earth's radius, that the angles a, b, c are very small, usually considerably below 1°; and for angles of that magnitude the logarithmic tables do not enable us to attain sufficient exactness. It consequently is attended with much less trouble, and with equal accuracy, to reduce the solution of such triangles as we have described to that of plane triangles, which may be effected by means of the following Proposition.

55. If the arcs which form the sides of a spherical triangle be very small relative to the radius of the sphere, then each of its angles will exceed the corresponding angle of the plane triangle whose sides are of the same length as the arcs forming the sides, by one-third of the spherical excess.

Let A', B', C' be the angles of the plane triangle whose sides are α, β, γ;

$$\text{then } \cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c};$$

$$\text{but } \cos a = \cos \frac{\alpha}{r} = 1 - \frac{\alpha^2}{2r^2} + \frac{\alpha^4}{2 . 3 . 4r^4}$$

$$\sin a = \sin \frac{\alpha}{r} = \frac{\alpha}{r} - \frac{\alpha^3}{2 . 3r^3},$$

expanding the cosine and sine in terms of the circular measure of the angle, and neglecting powers above the fourth; and similarly for b and c;

$$\therefore \cos A = \frac{\frac{1}{2r^2}(\beta^2 + \gamma^2 - \alpha^2) + \frac{1}{24r^4}(\alpha^4 - \beta^4 - \gamma^4 - 6\beta^2\gamma^2)}{\frac{\beta\gamma}{r^2}\left(1 - \frac{\beta^2 + \gamma^2}{6r^2}\right)}$$

$$= \frac{1}{2\beta\gamma}\left\{\beta^2+\gamma^2-\alpha^2+\frac{1}{12r^2}(\alpha^4-\beta^4-\gamma^4-6\beta^2\gamma^2)\right\}\left\{1+\frac{\beta^2+\gamma^2}{6r^2}\right\}$$

$$= \frac{\beta^2+\gamma^2-\alpha^2}{2\beta\gamma} + \frac{\alpha^4+\beta^4+\gamma^4-2\alpha^2\beta^2-2\alpha^2\gamma^2-2\beta^2\gamma^2}{24\beta\gamma r^2}$$

$$= \cos A' - \frac{\beta\gamma}{6r^2}\sin^2 A'.$$

Let $A = A' + \theta$; $\therefore \cos A = \cos A' - \theta \sin A'$;

$$\therefore \theta = \frac{\beta\gamma \sin A'}{6r^2} = \frac{S}{3r^2}, \text{ or } A = A' + \frac{S}{3r^2},$$

if S denote the area of the plane triangle; hence since S does not alter when A is interchanged with B or with C,

$$B = B' + \frac{S}{3r^2}, \quad C = C' + \frac{S}{3r^2};$$

$$\therefore A + B + C = \pi + \frac{S}{r^2} = \pi + E; \quad \therefore \frac{S}{r^2} = E,$$

and $A = A' + \frac{1}{3}E, \quad B = B' + \frac{1}{3}E, \quad C = C' + \frac{1}{3}E.$

56. Hence when three parts of the spherical triangle are given, we can always obtain three parts of the plane triangle, by means of which all the parts of the spherical triangle become known.

Suppose that the three arcs α, β, γ are given, then S the area of the plane triangle can be found at once, and therefore $E = \dfrac{S}{r^2 \sin 1''}$; if then A', B', C' be computed from α, β, γ, we shall by adding to each one third of E expressed in seconds, obtain the values of A, B, C.

Next suppose that A, β, γ are given, then $S = \frac{1}{2}\beta\gamma \sin A' = \frac{1}{2}\beta\gamma \sin A$ nearly, therefore $E = \dfrac{S}{r^2 \sin 1''}$ is known; hence in the plane triangle β, γ, and $A' = A - \frac{1}{3}E$ are known, and consequently its remaining parts, and therefore those of the spherical triangle, are known. If A, α, β are given, then

$\sin B' = \dfrac{\beta \sin A'}{\alpha} = \dfrac{\beta \sin A}{\alpha}$ nearly; therefore $C' = 180^\circ - A - B'$ nearly; therefore $S = \frac{1}{2}\, \alpha\beta \sin C'$, and $E = \dfrac{S}{r^2 \sin 1''}$.

If A, B, γ are given, then $S = \dfrac{\gamma^2 \sin A \sin B}{2 \sin (A + B)}$ nearly, and $E = \dfrac{S}{r^2 \sin 1''}$; and if A, B, α are given, then $C = C' = 180^\circ - A - B$ nearly, and we must proceed, as above, with the elements C, B, α.

57. To reduce an angle to the horizon.

Let BAC (fig. 12.) be an angle situated in an inclined plane, and having its vertex in the vertical line AD. Draw a horizontal plane meeting the lines AB, AC, AD in the points E, F, G; then the angle EGF is the horizontal projection of $\angle BAC$, or the $\angle BAC$ reduced to the horizon; and it is required to compute $\angle EGF$, supposing the angles BAC, BAD, CAD, determined instrumentally. Describe a sphere with center A and any radius, and let the planes EAF, FAG, GAE meet its surface in the arcs BC, CD, DB, forming the spherical triangle BCD; then its three sides are known, and the $\angle BDC = EGF$ is the angle sought, and may be computed from the formula

$$\sin \tfrac{1}{2} A = \sqrt{\frac{\sin (s - b) \sin (s - c)}{\sin b \sin c}},$$

where $\angle BAC = a$, $\angle BAD = b$, $\angle CAD = c$, and $s = \frac{1}{2}(a+b+c)$.

58. In practice, the angles b and c usually differ so little from right angles, that an approximate formula will suffice.

Let $b = \frac{1}{2}\pi - h$, $c = \frac{1}{2}\pi - h'$, then making $\sin h = h$, $\cos h = 1 - \frac{1}{2}h^2$, and supposing $A = a + \theta$, where θ is very small, we have

$$\cos (a + \theta) = \frac{\cos a - \sin h \sin h'}{\cos h \cos h'} = \frac{\cos a - hh'}{1 - \frac{1}{2}(h^2 + h'^2)};$$

$$\therefore \cos a - \theta \sin a = (\cos a - hh')\,(1 + \tfrac{1}{2}h^2 + \tfrac{1}{2}h'^2)$$

$$= \cos a + \tfrac{1}{2} \cos a\,(h^2 + h'^2) - hh';$$

$$\therefore\ \theta = hh' \operatorname{cosec} a - \tfrac{1}{2}(h^2 + h'^2)\cot a.$$

Let $p = \frac{1}{2}(h + h')$, $q = \frac{1}{2}(h - h')$;

$$\therefore p^2 - q^2 = hh',\ p^2 + q^2 = \tfrac{1}{2}(h^2 + h'^2);$$

$$\therefore \theta = (p^2 - q^2)\operatorname{cosec} a - (p^2 + q^2)\cot a = p^2 \tan\tfrac{1}{2}a - q^2 \cot\tfrac{1}{2}a.$$

59. Given two sides and the included angle of a spherical triangle, to find the angle between the chords of those sides.

Let ABC (fig. 15.) be a spherical triangle, O the center of the sphere, AB, AC, the chords of the arcs AB, AC, and a the angle included by them. Let pqr, be a spherical triangle described about A as center with any radius; then

$$\cos pq = \cos pr \cos qr + \sin pr \sin qr \cos prq.$$

But $\angle prq$ is the inclination of the planes AOB, AOC, and

$$= \angle A,\ \text{ and } pr = \angle BAO = 90^0 - \tfrac{1}{2}AOB = 90^0 - \tfrac{1}{2}c,$$

$$qr = \angle CAO = 90^0 = \tfrac{1}{2}AOC = 90^0 - \tfrac{1}{2}b;$$

$$\therefore \cos a = \sin\tfrac{1}{2}b \sin\tfrac{1}{2}c + \cos\tfrac{1}{2}b \cos\tfrac{1}{2}c \cos A.$$

Let $a = A - \theta$, where θ is usually very small; therefore, $\cos a = \cos A + \theta \sin A$; and the second member is the same as

$$\sin^2\tfrac{1}{4}(b + c) - \sin^2\tfrac{1}{4}(b - c) + \left\{\cos^2\tfrac{1}{4}(b+c) - \sin^2\tfrac{1}{4}(b-c)\right\}\cos A.$$

Hence, equating these values and reducing, we find

$$\theta = \sin^2\tfrac{1}{4}(b + c)\tan\tfrac{1}{2}A - \sin^2\tfrac{1}{4}(b - c)\cot\tfrac{1}{2}A, \text{ nearly}.$$

60. Given two sides and the included angle of a spherical triangle, to find the spherical excess.

$$-\cot\tfrac{1}{2}E = \tan\left\{\tfrac{1}{2}(A + B) + \tfrac{1}{2}C\right\} = \frac{\tan\frac{1}{2}(A+B) + \tan\frac{1}{2}C}{1 - \tan\frac{1}{2}(A+B)\tan\frac{1}{2}C}$$

$$= \frac{\dfrac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}\cot\tfrac{1}{2}C + \tan\tfrac{1}{2}C}{1 - \dfrac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}}$$

$$= \frac{\cos\frac{1}{2}(a-b)\dfrac{1+\cos C}{\sin C} + \cos\frac{1}{2}(a+b)\dfrac{1-\cos C}{\sin C}}{\cos\frac{1}{2}(a+b) - \cos\frac{1}{2}(a-b)}$$

$$= \frac{1}{\sin C} \cdot \frac{\cos\frac{1}{2}a\cos\frac{1}{2}b + \sin\frac{1}{2}a\sin\frac{1}{2}b\cos C}{-\sin\frac{1}{2}a\sin\frac{1}{2}b};$$

$$\therefore\ \cot\tfrac{1}{2}E = \cot\tfrac{1}{2}a\cot\tfrac{1}{2}b\operatorname{cosec} C + \cot C.$$

61. Given three sides of a spherical triangle, to find the spherical excess.

$$\tan\tfrac{1}{4}E = \tan\tfrac{1}{2}\left\{\tfrac{1}{2}(A+B) - \tfrac{1}{2}(180^\circ - C)\right\}$$

$$= \frac{\sin\frac{1}{2}(A+B) - \sin\frac{1}{2}(180^\circ - C)}{\cos\frac{1}{2}(A+B) + \cos\frac{1}{2}(180^\circ - C)}$$

$$= \frac{\cos\frac{1}{2}(a-b) - \cos\frac{1}{2}c}{\cos\frac{1}{2}(a+b) + \cos\frac{1}{2}c}\,\frac{\cos\frac{1}{2}C}{\sin\frac{1}{2}C},\ \text{(Art. 25.)}$$

$$= \frac{\sin\frac{1}{4}(a+c-b)\sin\frac{1}{4}(b+c-a)}{\cos\frac{1}{4}(a+b+c)\cos\frac{1}{4}(a+b-c)}\sqrt{\frac{\sin s\sin(s-c)}{\sin(s-a)\sin(s-b)}}$$

$$= \sqrt{\left\{\tan\tfrac{1}{2}s\tan\tfrac{1}{2}(s-a)\tan\tfrac{1}{2}(s-b)\tan\tfrac{1}{2}(s-c)\right\}}.$$

62. Having given the latitudes and longitudes of two places on the Earth's surface, to find their distance.

Let P be the pole (fig. 9.) GQR the equator, A, B, the two places in the meridians PQ, PR; PG the meridian of Greenwich; then the difference of longitude $= GR - GQ = QR = \angle APB = C$ suppose, and the colatitudes $PA = b$, $PB = a$, are known; so that in the spherical triangle APB there are given two sides and the included angle to find the third side $AB = c$, which may be done by the formulæ (Art. 43.)

$$\tan\theta = \tan a\cos C,\quad \cos c = \frac{\cos a\cos(b-\theta)}{\cos\theta};$$

then if D be the length of a quadrant of the meridian in miles, the distance of the places $= D\,.\,\dfrac{c}{90^\circ}$.

PROBLEMS.

1. In any triangle, to find the arc AD intercepted between a given point D in one of the sides, and the opposite angle, (fig. 8).

$$\cos AD = \cos AB \cos BD + \sin AB \sin BD \cos B,$$

and substituting for $\cos B$ its value in terms of the three sides, we get

$$\cos AD \sin BC = \cos AB \sin DC + \cos AC \sin BD.$$

2. On the surface of a sphere to draw a great circle passing through a given point and touching a given circle.

Let B be the given point, (fig. 16.), and P the pole of the given circle AC; then if BC be a great circle touching AC, and PC be joined, PCB is a right angle; and therefore $\cos BPC = \tan PC \cot PB$, which determines the point C, and consequently the circle BC.

3. If two arcs of great circles terminated by any circle, cut one another, the products of the tangents of the semi-segments are equal to one another.

Let AB, CD (fig. 17.) be the two arcs which intersect in F, P the pole of the circle in which they terminate; join PC, PA, PF, and draw the perpendiculars PH, PG. Then

$$\cos PG = \frac{\cos PF}{\cos FG} = \frac{\cos PA}{\cos AG}, \quad \cos PH = \frac{\cos PF}{\cos FH} = \frac{\cos PC}{\cos CH};$$

$$\therefore \frac{\cos CH}{\cos FH} = \frac{\cos AG}{\cos FG}, \quad \frac{\cos CH - \cos FH}{\cos CH + \cos FH} = \frac{\cos AG - \cos FG}{\cos AG + \cos FG},$$

$$\text{or} \quad \tan \tfrac{1}{2} AF \,.\, \tan \tfrac{1}{2} FB = \tan \tfrac{1}{2} CF \,.\, \tan \tfrac{1}{2} FD.$$

4. To prove that $\sin \frac{1}{2} E = \dfrac{\sin \frac{1}{2} a \sin \frac{1}{2} b}{\cos \frac{1}{2} c} \sin C$, where E is the spherical excess.

$$\sin\tfrac{1}{2}E = \sin\left\{\tfrac{1}{2}(A+B) - \tfrac{1}{2}(180^\circ - C)\right\}$$

$$= \sin\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}C - \cos\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}C,$$

$$= \left\{\cos\tfrac{1}{2}(a-b) - \cos\tfrac{1}{2}(a+b)\right\}\frac{\sin\frac{1}{2}C\cos\frac{1}{2}C}{\cos\frac{1}{2}c} \quad \text{(Art. 25.)}$$

$$= \frac{\sin\frac{1}{2}a\sin\frac{1}{2}b}{\cos\frac{1}{2}c}\sin C.$$

Hence, also, (Art. 60.)

$$\cos\tfrac{1}{2}E = \left\{\cos\tfrac{1}{2}a\cos\tfrac{1}{2}b + \sin\tfrac{1}{2}a\sin\tfrac{1}{2}b\cos C\right\}\sec\tfrac{1}{2}c.$$

5. If two arcs QBA, Qba, (fig. 18.), be intersected by two others PaA, PbB, in the points A, B, and a, b, then

$$\frac{\sin AQ}{\sin BQ} = \frac{\sin Aa}{\sin Pa}\,\frac{\sin Pb}{\sin Bb}.$$

For
$$\frac{\sin AQ}{\sin BQ} = \frac{\sin AQ}{\sin Aa}\cdot\frac{\sin Aa}{\sin Bb}\cdot\frac{\sin Bb}{\sin BQ}$$

$$= \frac{\sin a}{\sin Q}\cdot\frac{\sin Aa}{\sin Bb}\cdot\frac{\sin Q}{\sin b} = \frac{\sin Aa}{\sin Bb}\cdot\frac{\sin a}{\sin b}$$

$$= \frac{\sin Aa}{\sin Bb}\cdot\frac{\sin Pb}{\sin Pa}.$$

6. If two arcs ABC, abc, (fig. 18.) be intersected by three others which pass through the same point P, in the points A, B, C, and a, b, c, then

$$\frac{\sin Bb}{\sin Pb}\cdot\sin AC = \frac{\sin Aa}{\sin Pa}\sin BC + \frac{\sin Cc}{\sin Pc}\sin AB.$$

We have (p. 149.)*

$$\sin AC\cdot\sin BQ = \sin AQ\sin BC + \sin AB\sin CQ;$$

but
$$\frac{\sin AQ}{\sin CQ} = \frac{\sin Pc}{\sin Cc}\cdot\frac{\sin Aa}{\sin Pa},\quad \frac{\sin BQ}{\sin CQ} = \frac{\sin Pc}{\sin Cc}\,\frac{\sin Bb}{\sin Pb}.$$

Hence, eliminating $\sin BQ$ and $\sin AQ$ from the above equation, we get

$$\frac{\sin Bb}{\sin Pb}\cdot\sin AC = \frac{\sin Aa}{\sin Pa}\sin BC + \frac{\sin Cc}{\sin Pc}\sin AB.$$

* Trigon. Woodhouse.

This may be written

$$\left(\frac{\sin PB}{\tan Pb} - \cos PB\right) \sin AC = \left(\frac{\sin PA}{\tan Pa} - \cos PA\right) \sin BC$$

$$+ \left(\frac{\sin PC}{\tan Pc} - \cos PC\right) \sin AB; \text{ but (Prob. 1.),}$$

$$\cos PB \sin AC = \cos PA \, . \sin BC + \cos PC \sin AB,$$

$$\therefore \frac{\sin PB}{\tan Pb} . \sin AC = \frac{\sin PA}{\tan Pa} . \sin BC + \frac{\sin PC}{\tan Pc} . \sin AB.$$

If P be the pole of AC, then

$$\tan Bb . \sin AC = \tan Aa \, . \sin BC + \tan Cc . \sin AB,$$

which expresses the relation between the latitudes and longitudes of three places situated in the same great circle.

7. The arc passing through the middle points of the sides of any triangle upon a given base will meet the base produced, in a fixed point, whose distance from the middle point of the base is a quadrant.

We have (fig. 18.) $\dfrac{\sin AQ}{\sin CQ} = \dfrac{\sin Aa}{\sin Pa} . \dfrac{\sin Pc}{\sin Cc}$ (Prob. 5.),

if therefore a and c be the middle points of AP, CP, this becomes $\sin AQ = \sin CQ$; $\therefore AQ + CQ = 180^0$; and if B be the middle point of AC,

$$QB = \tfrac{1}{2}(AQ + CQ) = 90^0.$$

8. If one side of any triangle upon a given base be bisected by a secondary to the great circle which bisects the base at right angles, the other side will also be bisected by it.

For if $QB = 90^0$, then $AQ + QC = 180^0$, and $\sin AQ = \sin CQ$;

$$\therefore \frac{\sin Aa}{\sin Pa} = \frac{\sin Cc}{\sin Pc}, \quad \text{and if } Aa = Pa, \quad \text{then } Cc = Pc.$$

9. All triangles upon a given base, and which have the middle points of their other two sides in the same great circle, are equal in area. (Fig. 19.)

Let AB be the given base, D its middle point, $DK = 90^0$, KFG any circle passing through K; then if ACB be a triangle

having one of its sides AC bisected in G, the other will be bisected in F (Prob. 8.); and if E be its spherical excess,

$$\cot \tfrac{1}{2} E = \frac{\cot \frac{1}{2} a \cot \frac{1}{2} c + \cos B}{\sin B} = \cot K \operatorname{cosec} \tfrac{1}{2} c, \text{ from the}$$

triangle KBF (Art. 22.), since $KD = 90^0$; $\therefore$ E is invariable. Also

$$\cos FG = \cos \tfrac{1}{2} a \cos \tfrac{1}{2} b + \sin \tfrac{1}{2} a \sin \tfrac{1}{2} b \cos C = \cos \tfrac{1}{2} E \cos \tfrac{1}{2} c,$$

(Prob. 4.), therefore the distance of the middle points of the sides of all the triangles is the same.

10. Upon a given base to construct a triangle which shall have a given angle at the base, and shall be equal in area to a given triangle upon the same base.

AB (fig. 20.) the given base, and ACB the given triangle, GF the arc joining the middle points of its sides.

Make $BAG' =$ the given angle, take $G'F' = GF$, and produce BF' to meet AG' in C'; then ABC' is the triangle required (Prob. 9).

11. To construct a triangle which shall be equal in area to a given triangle, and have an angle in common with it, and have one of the sides containing that angle of a given length.

ACB the given triangle (fig. 21.) and C the given angle. Take CB' equal to the given side, bisect BB' in F, and AB in G; produce FG to meet CA in F', make $F'A' = FA$, and join $A'B'$; then $A'CB'$ is the required triangle. For the triangles $A'AB'$, ABB', are evidently equal to one another. (Prob. 9).

12. To find the locus of the vertex of a spherical triangle of given base and area.

AB the given base (fig. 19.), D its middle point, $Dp = x$, $pC = y$ the spherical co-ordinates of the vertex of the triangle whose area is E, α and β the areas of the right-angled triangles ACp, BCp, respectively;

then $\cot \tfrac{1}{2} \alpha = \cot (\tfrac{1}{4} c + \tfrac{1}{2} x) \cot \tfrac{1}{2} y$, (Art. 60.)

$\cot \tfrac{1}{2} \beta = \cot (\tfrac{1}{4} c - \tfrac{1}{2} x) \cot \tfrac{1}{2} y$;

$$\therefore \cot\tfrac{1}{2}\alpha \,.\, \cot\tfrac{1}{2}\beta = \frac{\cos x + \cos\frac{1}{2}c}{\cos x - \cos\frac{1}{2}c}\cot^2\tfrac{1}{2}y, \text{ (p. 34 Pl. Trig.),}$$

$$\cot\tfrac{1}{2}\alpha + \cot\tfrac{1}{2}\beta = \frac{2\sin\frac{1}{2}c}{\cos x - \cos\frac{1}{2}c}\cot\tfrac{1}{2}y, \quad \text{(p. 36);}$$

$$\therefore \cot\tfrac{1}{2}E = \frac{\cot\frac{1}{2}\alpha\cot\frac{1}{2}\beta - 1}{\cot\frac{1}{2}\alpha + \cot\frac{1}{2}\beta} = \frac{\cos\frac{1}{2}c\operatorname{cosec}^2\frac{1}{2}y + \cos x\,(\cot^2\frac{1}{2}y - 1)}{2\sin\frac{1}{2}c\cot\frac{1}{2}y}$$

$$= \cot\tfrac{1}{2}c \,.\, \operatorname{cosec} y + \cos x \operatorname{cosec}\tfrac{1}{2}c\cot y;$$

$$\text{or} \quad \cos\tfrac{1}{2}c = \sin\tfrac{1}{2}c\cot\tfrac{1}{2}E\sin y - \cos x\cos y,$$

the equation to the required locus. But if x', y', be the co-ordinates reckoned from D of the pole, and r the angular radius of a circle of the sphere, the equation to its perimeter is evidently

$$\cos r = \sin y'\sin y + \cos y'\cos y\cos(x - x'),$$

which coincides with the above if

$$x' = 0, \quad -\tan y' = \sin\tfrac{1}{2}c\cot\tfrac{1}{2}E, \quad -\frac{\cos r}{\cos y'} = \cos\tfrac{1}{2}c,$$

which determine x', y', r. The first shews that the center M of the locus lies in the secondary DD' to the base through its middle point; the second that $y' = \angle K + 90^\circ = DE + 90^\circ$, GEF being an arc joining the middle points of the sides AC, BC, (Prob. 9.); the third that if AC, and BC, be produced to meet the circle ADB in A', B', then A', B', are points in the locus; for $\cos MD' = -\cos y'$, and $A'D' = \frac{1}{2}c$, $\therefore \cos r = \cos MD'\cos A'D'$;

$$\therefore r = MA' = MB'.$$

13. To find the angular radius of the circle which passes through the angular points of a given triangle, in terms of its angles, or sides.

Draw the arcs aO, bO, (fig. 22.), perpendicular to the sides BC, AC, through their middle points, and draw Oc perpendicular to AB. Join OA, OB, OC; then O is evidently the pole of the circumscribed circle, and c the middle point of AB. Hence $\angle C + OAB = \frac{1}{2}(A + B + C)$, or $OAB = S - C$; but from the right-angled triangle OAc, $\cos OAc = \cot OA\tan Ac$,

$$\therefore \tan R = \frac{\tan\frac{1}{2}c}{\cos(S - C)}.$$

Also, $\cos(S-C) = \cos\frac{1}{2}(A+B)\cos\frac{1}{2}C + \sin\frac{1}{2}(A+B)\sin\frac{1}{2}C$,

$$= \left\{\cos\tfrac{1}{2}(a+b) + \cos\tfrac{1}{2}(a-b)\right\}\frac{\sin\frac{1}{2}C\cos\frac{1}{2}C}{\cos\frac{1}{2}c} = \frac{\cos\frac{1}{2}a\cos\frac{1}{2}b\sin C}{\cos\frac{1}{2}c},$$

$$\tan R = \frac{\sin\frac{1}{2}c}{\cos\frac{1}{2}a\cos\frac{1}{2}b\sin C};$$

and if for $\tan\frac{1}{2}c$, $\sin C$, we substitute their values (Arts. 48 and 36), we get $\tan R$ expressed in terms of the angles and sides respectively.

14. If O be the pole of the circle circumscribing a triangle ABC, then

$$\cos\tfrac{1}{2}AOB = \cos\tfrac{1}{2}a\cos\tfrac{1}{2}b\cos C + \sin\tfrac{1}{2}a\sin\tfrac{1}{2}b.$$

This results from Art. 59, for it is evident that the angle between the chords of AC and BC is half the spherical angle AOB.

15. To find the angular radius of the circle which touches the three sides of a given triangle, in terms of the sides, or angles, of the triangle.

Bisect the angles A and B by the arcs of AO, BO, (fig. 22.) join CO, and from O draw the arcs Oa, Ob, Oc perpendicular to the sides, or sides produced; then these perpendiculars are evidently equal to one another, therefore O is the pole of the required circle, and angle C is bisected by OC; also $Ac = Ab$, $Ba = Bc$ and $Ca = Cb$. Hence, when the circle is inscribed, $AB + Cb = \frac{1}{2}(a+b+c)$, or $Cb = s - c$; and when the circle touches the side c and the two others produced, $Cb = s$; but from the right-angled triangle CbO, $\sin Cb = \tan Ob \,.\, \cot\frac{1}{2}C$;

therefore, for the inscribed circle, $\tan r = \sin(s-c)\tan\frac{1}{2}C$;

and for the circle which touches the side c, and the two others produced, $\tan r' = \sin s\tan\frac{1}{2}C$.

Also $\tan r = \left\{\sin\frac{1}{2}(a+b)\cos\frac{1}{2}c - \cos\frac{1}{2}(a+b)\sin\frac{1}{2}c\right\}\tan\frac{1}{2}C$,

$$= \left\{\cos\tfrac{1}{2}(A-B) - \cos\tfrac{1}{2}(A+B)\right\}\frac{\sin\frac{1}{2}c\cos\frac{1}{2}c}{\cos\frac{1}{2}C},$$

$$= \frac{\sin\frac{1}{2}A\sin\frac{1}{2}B}{\cos\frac{1}{2}C}\,.\,\sin c;\text{ and similarly,}$$

$$\tan r' = \frac{\cos\frac{1}{2}A\cos\frac{1}{2}B}{\cos\frac{1}{2}C}\sin c;$$

and if for $\tan \frac{1}{2} C$, and $\sin c$, we substitute their values, we get $\tan r$ and $\tan r'$ expressed in terms of the sides, and angles, respectively.

16. If three arcs be drawn from the angles of a triangle through any point P, to meet the opposite sides in A', B', C', and M be the pole of the circumscribed circle, then

$$\frac{\sin PA'}{\sin AA'} + \frac{\sin PB'}{\sin BB'} + \frac{\sin PC'}{\sin CC'} = \frac{\cos PM}{\cos R}.$$

Let O (fig. 23.) be the center of the sphere, and let the radii OP, OM, OB', OA', meet the plane of the circumscribed circle in p, m, b, a; then $\angle\ Omp = 90^0$, and

$$\frac{pb}{Bb} = \frac{Op \sin PB'}{OB \sin BB'} = \frac{Op}{Om} \cdot \frac{Om}{OB} \cdot \frac{\sin PB'}{\sin BB'} = \frac{\cos BM}{\cos PM} \cdot \frac{\sin PB'}{\sin BB'};$$

$$\therefore\ \frac{pa}{Aa} + \frac{pb}{Bb} + \frac{pc}{Cc} =$$

$$1 = \left\{\frac{\sin PA'}{\sin AA'} + \frac{\sin PB'}{\sin BB'} + \frac{\sin PC'}{\sin CC'}\right\} \cdot \frac{\cos BM}{\cos PM}.$$

17. To find the angular distance between the poles of the inscribed and circumscribed circles.

Suppose P (fig. 23) to be the pole of the inscribed circle; then from the right-angled triangle PNB'

$$\sin r = \sin PB' \sin B' = \frac{\sin PB'}{\sin BB'} \sin C \sin a,$$

$$\therefore \frac{\cos D}{\cos R \sin r} = \frac{1}{\sin B \sin c} + \frac{1}{\sin C \sin a} + \frac{1}{\sin A \sin b} = \frac{\sin a + \sin b + \sin c}{M},$$

where $M = \sin b \sin c \sin A$

$$= \sqrt{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c},$$

$$\text{and } \therefore \cot r = \frac{2 \sin s}{M}, \quad \tan R = \frac{4 \sin \frac{1}{2} a \sin \frac{1}{2} b \sin \frac{1}{2} c}{M};$$

$$\therefore \left(\frac{\cos D}{\cos R \sin r}\right)^2 - 1$$

$$= \frac{2}{M^2}(1 + \sin a \sin b + \sin a \sin c + \sin b \sin c - \cos a \cos b \cos c)$$

$$= \frac{1}{M^2}\left\{2 \sin \tfrac{1}{2}(a+b) \cos \tfrac{1}{2}c + 2 \cos \tfrac{1}{2}(a-b) \sin \tfrac{1}{2}c\right\}^2 \text{ (p. 149.)}$$

$$= \frac{1}{M^2}\left\{2 \sin s + 4 \sin \tfrac{1}{2}a \sin \tfrac{1}{2}b \sin \tfrac{1}{2}c\right\}^2$$

$$= \left\{\cot r + \tan R\right\}^2;$$

$$\therefore\ \sin^2 D = \sin^2 (R - r) - \cos^2 R \sin^2 r.$$

If the circle touch the side c and the two others produced, then it will be similarly found that

$$\sin^2 D = \sin^2 (R + r') - \cos^2 R \sin^2 r'.$$

18. If three arcs be drawn from the angles of a triangle through any point to meet the opposite sides, then the products of the sines of the alternate segments of the sides are equal to one another.

By Prob. 5. we have, (fig. 23.)

$$\frac{\sin BC'}{\sin BA} = \frac{\sin CB'}{\sin AB'} \cdot \frac{\sin C'P}{\sin CP}, \quad \frac{\sin AC'}{\sin AB} = \frac{\sin CA'}{\sin BA'} \cdot \frac{\sin C'P}{\sin CP};$$

$$\therefore\ \frac{\sin AC'}{\sin BC'} = \frac{\sin AB' \cdot \sin CA'}{\sin CB' \cdot \sin BA'}.$$

Conversely, when the points A', B', C' in the sides, satisfy this condition, the arcs joining them with the opposite angles, intersect one another in the same point. The cases in which this condition is fulfilled are exactly the same as for plane triangles. If A', B', C' be the feet of the perpendiculars, the condition is evidently fulfilled, since we have from the properties of right-angled triangles,

$$\cos AC' \cos CB' \cos BA' = \cos BC' \cos AB' \cos CA',$$
$$\tan AC' \tan CB' \tan BA' = \tan BC' \tan AB' \tan CA'.$$

19. The product of the sines of the semi-diagonals of a quadrilateral inscribed in a circle, is equal to the sum of the products of the sines of half the opposite sides.

Let the dotted lines (fig. 24.) represent the chords of the arcs; then they all lie in the plane of the circle circumscribing the quadrilateral, and

$$AD \,.\, BC = AB \,.\, CD + AC \,.\, BD\,;$$

$$\therefore \sin\tfrac{1}{2}AD \,.\, \sin\tfrac{1}{2}BC = \sin\tfrac{1}{2}AB \,.\, \sin\tfrac{1}{2}CD + \sin\tfrac{1}{2}AC \,.\, \sin\tfrac{1}{2}BD.$$

It is easily seen that the sums of the two opposite angles of the quadrilateral are equal to one another.

20. If S be the number of the solid angles of a polyhedron, F the number of its faces, E the number of its edges, then $S + F = E + 2$.

From any internal point as a center, suppose a sphere to be described with radius 1; join the center with each of the angular points of the polyhedron, and then join all the points where these lines meet the surface of the sphere by arcs of great circles; there will thus be formed as many polygons as there are faces.

If therefore s be the sum of the angles of any one of these polygons, and n the number of its sides, (since it may be divided into as many triangles as it has sides, having a common vertex) its area $= s + 2\pi - n\pi$; and therefore, adding the areas of all the polygons together of which the number is F,

$$\text{area of sphere} = 4\pi = \Sigma(s) + 2F\pi - \pi\Sigma(n).$$

But $\Sigma(s)$ = sum of all the angles of all the polygons $= 2\pi \times$ number of solid angles $= 2\pi S$; and $\Sigma(n)$ = number of all the sides of all the polygons = twice the number of edges $= 2E$, because each edge gives an arc common to two polygons;

$$\therefore\ 4\pi = 2\pi S + 2\pi F - 2\pi E, \text{ or } S + F = E + 2.$$

21. There can be only five regular polyhedrons.

In the case of a regular polyhedron, every face has the same number (n) of sides, and every solid angle the same number (m) of faces; and the entire number of plane angles in all the faces is equally expressed by nF or mS or $2E$;

$$\therefore\ nF = mS = 2E, \text{ and } S + F = E + 2; \text{ hence}$$

$$S = \frac{4n}{2(m+n) - mn}, \text{ which must be a positive integer;}$$

$\therefore \frac{1}{m} + \frac{1}{n} > \frac{1}{2}$; but the greatest value both of $\frac{1}{m}$ and $\frac{1}{n}$ is $\frac{1}{3}$, therefore neither $\frac{1}{m}$ nor $\frac{1}{n}$ can be so small as $\frac{1}{6}$; therefore the only admissible values for m and n are 3, 4, 5; and those combinations of them must be taken which make S, F, and E integers. It will be found that if the faces are equilateral triangles, or $n = 3$, we may form each solid angle of the polyhedron with 3, 4, or 5 angles of these triangles, and so form the tetrahedron, the octahedron, and the icosahedron, or the solids of four, eight, and twenty faces, respectively. If the faces are squares, or $n = 4$, we may form each solid angle with three plane angles, and so form the cube or hexahedron; and if the faces are pentagons, or $n = 5$, by uniting three of their angles to form a solid angle, we obtain the dodecahedron; and these are all the regular polyhedrons that can exist.

22. To find the inclination of two adjacent faces of a regular polyhedron, and the radii of the inscribed and circumscribed spheres.

Let $AB = a$ (fig. 25.) represent an edge of the polyhedron common to two adjacent faces whose centers are C and E; $CO = r$ the radius of the inscribed, $AO = R$ that of the circumscribed sphere; OD perpendicular to AB. Let the planes AOC, COD, AOD, meet the surface of a sphere whose center is O, in the arcs pq, qr, rp; then $\angle q = \frac{1}{2} \cdot \frac{2\pi}{n}$, $\angle p = \frac{2\pi}{2m}$, and $\angle r = \frac{1}{2}\pi$; $\therefore \cos p = \cos qr \cdot \sin q$; but $\cos qr = \cos COD$

$$= \sin CDO = \sin \tfrac{1}{2} I, \text{ if } I = \angle CDE; \ \therefore \sin \tfrac{1}{2} I = \frac{\cos \frac{\pi}{m}}{\sin \frac{\pi}{n}}.$$

Also $\cos pq = \cot p \cdot \cot q$, or $\frac{R}{r} = \tan \frac{\pi}{m} \tan \frac{\pi}{n}$,

and $R^2 = r^2 + \frac{1}{4} a^2 \operatorname{cosec}^2 \frac{\pi}{n}$, from which equations R and r may be found.

The area of each face $= \frac{n}{4} a^2 \cot \frac{\pi}{n}$;

$\therefore$ area of surface of polyhedron $= \frac{nF}{4} a^2 \cot \frac{\pi}{n}$,

and volume $= \frac{1}{3}$ area of surface $\times$ radius of inscribed sphere.

Also the radius of the sphere to which the edges are tangents $OD = \sqrt{R^2 - \frac{1}{4}a^2}$; and that to which one face and the n adjacent ones produced, are tangents,

$$= CD \cot \tfrac{1}{2} I = \tfrac{1}{2} a \cot \frac{\pi}{n} \cot \tfrac{1}{2} I.$$

23. To find the volume of a parallelopiped in terms of its edges and their inclinations to one another.

Let the edges be $SA = a$, $SB = b$, $SC = c$, (fig. 27.), and the angles which they make with one another $BSC = \alpha$, $ASC = \beta$, $ASB = \gamma$; drop the perpendicular CO from C on the plane ASB, and let arc FG be the intersection of the plane CSO with the surface of a sphere whose center is S, and DE, EF, FD, the intersections of the faces of the parallelopiped with the same sphere. Then volume of parallelopiped $=$ area of base $AB \times$ altitude $CO = ab \sin \gamma \times c \sin FG$

$= abc \sin \gamma \times \sin \alpha \sin E$, from the right-angled triangle FEG,

$= abc \sqrt{1 - \cos^2 \alpha - \cos^2 \beta - \cos^2 \gamma + 2 \cos \alpha \cos \beta \cos \gamma}$.

24. To find the diagonal of a parallelopiped in terms of its edges, and their inclinations to one another.

Let FH be the intersection of the plane SPT with the sphere whose center is S; then

$$ST^2 = SP^2 + PT^2 + 2SP \,.\, PT \cos FH$$

$$= a^2 + b^2 + 2ab\cos\gamma + c^2 + 2c \,.\, \frac{SP}{\sin\gamma} \{\cos\alpha \,.\, \sin HD + \cos\beta \sin HE\}$$

$$= a^2 + b^2 + c^2 + 2ab \cos\gamma + 2c\,\{b\cos\alpha + a\cos\beta\}, \text{ (Prob. 1.)}$$

25. Given the six edges of any tetrahedron, to find its volume.

Let $SABC$ (fig. 27.) be the tetrahedron; then it will evidently be one-third of the prism whose base is ASB and

height CO, and consequently one-sixth of the parallelopiped whose edges are SA, SB, SC, and therefore its volume

$$= \tfrac{1}{6} abc \sqrt{1 - \cos^2 \alpha - \cos^2 \beta - \cos^2 \gamma + 2 \cos \alpha \cos \beta \cos \gamma};$$

but if $AB = c'$, $AC = b'$, $BC = a'$, then $2ab \cos \gamma = a^2 + b^2 - c'^2$, $2ac \cos \beta = a^2 + c^2 - b'^2$, $2bc \cos \alpha = b^2 + c^2 - a'^2$; and these values of the cosines may be substituted.

26. To find the radii r and R of the inscribed and circumscribed spheres of any tetrahedron.

First we have $\frac{1}{3} r \times$ area of surface = volume.

Also the center of the circumscribed sphere will evidently be the intersection of two lines at right angles to two faces, passing through the centers of their circumscribed circles. Let EDF (fig. 28.) be a plane bisecting the edge SC at right angles, and therefore containing the perpendiculars EO, FO, to the faces ACS, BSC, through the centers of the circles which circumscribe them, and consequently O the center of the circumscribed sphere. Then

$$SO^2 = SD^2 + DO^2 = SD^2 + \left(\frac{FE}{\sin D}\right)^2$$

$$= SD^2 + \frac{1}{\sin^2 D}(ED^2 + DF^2 - 2ED \,.\, DF \cos D);$$

$$\text{where } SD = \tfrac{1}{2}c, \quad ED = \frac{a - c\cos\beta}{2\sin\beta}, \quad DF = \frac{b - c\cos\alpha}{2\sin\alpha},$$

$$\text{and } \cos D = \frac{\cos\gamma - \cos\alpha\cos\beta}{\sin\alpha\sin\beta}.$$

Or, if the length of an edge, the inclination of the faces which intersect in it, and the angles which it subtends at the other vertices of the tetrahedron be given, then

$$SO^2 = SD^2 + \frac{SD^2}{\sin^2 D}(\cot^2 A + \cot^2 B - 2\cot A \cot B \cos D).$$

27. To determine the arc which joins the vertices of two given triangles upon the same base.

Suppose the base bisected in O (fig. 20.); then (Prob. 1.) $2 \cos \frac{1}{2}c \cos CO = \cos a + \cos b$; and if p be the perpendicular

from C on AB, $\sin c \sin p = \sin a \sin b \sin C$; and similarly for the triangle $AC'B$. Hence

$$\cos CC' = \cos CO \cos C'O + \sin CO \sin C'O \cos (AOC' - AOC)$$

but

$$\sin CO \sin \tfrac{1}{2} c \cos AOC = \cos b - \cos \tfrac{1}{2} c \cos CO = \tfrac{1}{2} (\cos b - \cos a),$$

$$\sin CO \sin AOC = \sin p;$$

$$\therefore \sin^2 c \cos CC' = (\cos a + \cos b)(\cos a' + \cos b') \sin^2 \tfrac{1}{2} c + (\cos b - \cos a)$$

$$\times (\cos b' - \cos a') \cos^2 \tfrac{1}{2} c + \sin a \sin b \sin C \sin a' \sin b' \sin C';$$

which will contain only the sides, if for $\sin C$, $\sin C'$, their values be substituted. This is the Problem of finding the latitude from two altitudes of the Sun and the time between.

28. If M be the pole of the circle circumscribing an equilateral triangle ABC, and P any point on the sphere, then (fig. 23.) $\cos PA + \cos PB + \cos PC = 3 \cos R \cos PM$.

$$\cos PA = \cos R \cos D + \sin R \sin D \cos AMP,$$

$$\cos PB = \cos R \cos D + \sin R \sin D \cos (120^0 - AMP),$$

$$\cos PC = \cos R \cos D + \sin R \sin D \cos (120^0 + AMP);$$

$$\therefore \cos PA + \cos PB + \cos PC = 3 \cos R \cos D.$$

29. If in a spherical triangle the angle C and the opposite side c remain constant, to find the relations between the corresponding small variations of any two of the other parts.

First for δa and δb the variations of the sides a and b. Writing down, in this and all similar cases, the formula involving the constant elements, and the two whose variations are to be compared, we have

$$\cos c = \cos a \cos b + \sin a \sin b \cos C,$$

$$\therefore \cos c = \cos (a+\delta a) \cos (b+\delta b) + \sin (a+\delta a) \sin (b+\delta b) \cos C;$$

or since δa is very small, and therefore $\sin \delta a = \delta a$, $\cos \delta a = 1$, and the same for δb,

$$\cos c = (\cos a - \sin a \delta a)(\cos b - \sin b \delta b) + (\sin a + \cos a \delta a)$$

$$\times (\sin b + \cos b \delta b) \cos C;$$

hence, subtracting, and neglecting terms involving $\delta a \delta b$,

$$0 = \delta a(\sin a \cos b - \cos a \sin b \cos C) + \delta b(\sin b \cos a - \cos b \sin a \cos C),$$

or, dividing by $\sin a \sin b$, we get (Art. 22.),

$$0 = \frac{\delta a}{\sin a} \cot B \sin C + \frac{\delta b}{\sin b} \cot A \sin C; \ \therefore \delta a \cos B + \delta b \cos A = 0.$$

Hence from the polar triangle, if δA, δB be the variations of the angles, whilst C and c remain unchanged,

$$\delta A \cos b + \delta B \cos a = 0.$$

The above result may be also obtained geometrically as follows.

Let ACB (fig. 26.) be the triangle, and Ca, Cb, the altered values of the sides, so that $ab = AB$; draw the arcs aa, $B\beta$ perpendicular to AO, bO, then $a\beta = Ba$ nearly, and therefore $Aa = b\beta$, or, since the triangles Aaa, $Bb\beta$ may be regarded as plane triangles,

$$Aa \cos A = Bb \cos b, \text{ or } \delta a \cos B + \delta b \cos A = 0, \text{ nearly.}$$

To compare the variations of A and a, or of A and b, we must proceed in the same way with the equations

$$\sin A \sin c = \sin C \sin a,$$

$$\cot C \sin A = \cot c \sin b - \cos A \cos b;$$

and we shall find

$$\cot A \delta A = \cot a \delta a, \quad \cos B \delta A = -\cot a \sin A \delta b.$$

30. To compare the corresponding small variations of any two of the other parts of a triangle, supposing two angles B, C, or two sides b, c, to remain constant.

First, suppose B and C to remain constant; then proceeding as above with the equations

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a,$$

$$\sin b \sin C = \sin c \sin B$$

$$\cos B = -\cos A \cos C + \sin A \sin C \cos b,$$

$$\cot B \sin C = \cot b \sin a - \cos C \cos a,$$

we find

$$\delta A = \sin b \sin C \delta a, \qquad \cot b \delta b = \cot c \delta c, \qquad \sin A \delta b = \cot c \delta A,$$

$$\sin A \delta b = \sin B \cos c \delta a.$$

Hence by the Polar Triangle, supposing the two sides b and c to remain constant, we have

$$\delta a = \sin B \sin c \delta A, \qquad \cot B \delta B = \cot C \delta C,$$

$$\sin a \delta B = -\cot C \delta a, \qquad \sin a \delta B = -\sin b \cos C \delta A.$$

31. To compare the corresponding small variations of any two of the other parts of a triangle, supposing an angle and a side adjacent to it, A, c, to remain constant.

$$\delta a = \cos C \delta b, \qquad \tan a \delta C = -\tan C \delta a, \qquad \delta C = -\cos a \delta B,$$

$$\sin a \delta B = \sin C \delta b, \qquad \tan a \delta C = -\sin C \delta b,$$

$$\sin a \delta B = \tan C \delta a.$$

32. From any formula in Spherical Trigonometry involving the parts of a triangle, one of them being a side, to deduce the corresponding formula in Plane Trigonometry.

Since the quantities involved are the angles of inclination of the faces and edges of a solid angle expressed by their circular measures, if we suppose α, β, γ to be the lengths of the arcs in which the planes of the faces cut the surface of a sphere described from the vertex of the solid angle with any radius r, we shall have $\alpha = ar$, $\beta = br$, $\gamma = cr$; and if we substitute for a, b, c, these values in the proposed formula, and then suppose r to be very large so that $\frac{a}{r}, \frac{\beta}{r}, \frac{\gamma}{r}$, become very small, and therefore $\sin \frac{a}{r}$, $\cos \frac{a}{r}$, &c. may be replaced by one or two terms of their expansions, we obtain a nearly exact relation between the lengths of the arcs upon a sphere whose radius is r, and the angles of inclination of the planes in which the arcs lie; and if we now make r infinite, we obtain an exact relation between the sides and angles of a plane triangle. Thus, as at Art. 55, we have

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c} = \frac{\cos\frac{a}{r} - \cos\frac{\beta}{r}\cos\frac{\gamma}{r}}{\sin\frac{\beta}{r}\sin\frac{\gamma}{r}}$$

$$= \frac{\left(1 - \frac{1}{2}\frac{a^2}{r^2} + \frac{1}{24}\frac{a^4}{r^4}\right) - \left(1 - \frac{1}{2}\frac{\beta^2}{r^2}\right)\left(1 - \frac{1}{2}\frac{\gamma}{r^2}\right)}{\frac{\beta}{r}\cdot\frac{\gamma}{r}}$$

$$= \frac{\beta^2 + \gamma^2 - a^2}{2\beta\gamma} + \frac{a^4 + \&c.}{24\beta\gamma r^2} \text{ nearly, supposing } r \text{ very large;}$$

$\therefore$ making r infinite, we get for a plane triangle whose sides are a, β, γ,

$$\cos A = \frac{\beta^2 + \gamma^2 - a^2}{2\beta\gamma}.$$


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